Oct. 5 Math 1190 sec. 52 Fall 2016

Section 3.3: Derivatives of Logarithmic Functions

Properties of logarithms can be used to simplify expressions characterized by products, quotients and powers.

Illustrative Example: Evaluate $$\frac{d}{dx}\ln\left(\frac{x^{2/3}(x+3)^5}{\sin^{-1}x}\right)$$

We'll use log properties first.

$$\ln\left(\frac{x^{2/3}(x+3)^5}{\sin^{-1}x}\right) = \ln\left(x^{2/3}(x+3)^5\right) - \ln\left(\sin^{-1}x\right)$$

$$= \ln x^{2/3} + \ln(x+3)^5 - \ln\left(\sin^{-1}x\right)$$

$$= \frac{2}{3}\ln x + 5\ln(x+3) - \ln(\sin^{-1}x)$$

* Recall $\sin^{-1}x \neq (\sin x)^{-1}$ it's $\arcsin(x)$

Now we take the derivative

$$\frac{d}{dx}\ln\left(\frac{x^{2/3}(x+3)^5}{\sin^{-1}x}\right) = \frac{d}{dx}\left(\frac{2}{3}\ln x + 5\ln(x+3) - \ln(\sin^{-1}x)\right)$$

$$= \frac{2}{3}\frac{1}{x} + 5\frac{1}{x+3} - \frac{\frac{1}{\sqrt{1-x^2}}}{\sin^{-1}x}$$

$$= \frac{2}{3x} + \frac{5}{x+3} - \frac{1}{\sin^{-1}x\sqrt{1-x^2}}$$

$$\frac{d}{dx} \ln f(x) = \frac{f'(x)}{f(x)}$$

Question

Evaluate the derivative. Use properties of logs to simplify the process.

$$\frac{d}{dx} \ln \left(\frac{\sqrt{x}}{\tan x + 1} \right)$$

Which expression is the correct derivative

(a) $\frac{1}{2x} - \frac{\sec^2 x}{\tan x}$

(b) $\frac{1}{2x} - \frac{\sec^2 x}{\tan x + 1}$

(c) $\frac{x}{2} - \tan x - 1$

(d) $\frac{1}{\frac{1}{2}x^{-1/2}} - \frac{1}{\sec^2 x}$

$$\ln\left(\frac{\sqrt{x}}{\tan x + 1}\right) = \ln\sqrt{x} - \ln(\tan x + 1)$$

$$= \frac{1}{2}\ln x - \ln(\tan x + 1)$$

$$\frac{d}{dx}\ln\left(\frac{\sqrt{x}}{\tan x + 1}\right) = \frac{d}{dx}\left(\frac{1}{2}\ln x - \ln(\tan x + 1)\right)$$

$$= \frac{1}{2} \cdot \frac{1}{x} - \frac{\sec^2 x}{\tan x + 1}$$

Logarithmic Differentiation

We can use properties of logarithms to simplify the process of taking derivatives of expressions that are complicated by

products quotients and powers.

Illustrative Example: Evaluate $\dfrac{d}{dx}\left(\dfrac{x^2\sqrt{x+1}}{\cos^4(3x)}\right)$

If y is a differentiable function of x, then by the chain rule

$$\frac{d}{dx}\ln y = \frac{1}{y}\frac{dy}{dx}$$

$$\Rightarrow \quad \frac{dy}{dx} = y\left(\frac{d}{dx}\ln y\right)$$

This is useful if computing $\frac{d}{dx}\ln y$ is easier than computing $\frac{dy}{dx}$ directly.

Let $y = \dfrac{x^2\sqrt{x+1}}{\cos^4(3x)}$. Take the log.

$$\ln y = \ln\left(\frac{x^2\sqrt{x+1}}{\cos^4(3x)}\right) = \ln\left(x^2\sqrt{x+1}\right) - \ln\left(\cos(3x)\right)^4$$

$$= \ln x^2 + \ln(x+1)^{1/2} - \ln\left(\cos(3x)\right)^4$$

$$= 2\ln x + \frac{1}{2}\ln(x+1) - 4\ln\cos(3x)$$

Now take the derivative.

$$\frac{d}{dx} \ln y = \frac{d}{dx}\left(2\ln x + \frac{1}{2}\ln(x+1) - 4\ln \cos(3x)\right)$$

$$\frac{1}{y}\frac{dy}{dx} = 2\frac{1}{x} + \frac{1}{2}\frac{1}{x+1} - 4\frac{-\sin(3x)\cdot 3}{\cos(3x)}$$

$$= \frac{2}{x} + \frac{1}{2(x+1)} + 12\frac{\sin(3x)}{\cos(3x)}$$

$$\frac{dy}{dx} = y\left(\frac{2}{x} + \frac{1}{2(x+1)} + 12\tan(3x)\right)$$

Now use $y = \dfrac{x^2\sqrt{x+1}}{\cos^4(3x)}$ to get

$$\frac{dy}{dx} = \frac{x^2\sqrt{x+1}}{\cos^4(3x)}\left(\frac{2}{x} + \frac{1}{2(x+1)} + 12\tan(3x)\right)$$

Logarithmic Differentiation

If the differentiable function $y = f(x)$ consists of complicated products, quotients, and powers:

(i) Take the logarithm of both sides, i.e. $\ln(y) = \ln(f(x))$. Then use properties of logs to express $\ln(f(x))$ as a sum/difference of simpler terms.

(ii) Take the derivative of each side, and use the fact that $\frac{d}{dx}\ln(y) = \frac{\frac{dy}{dx}}{y}$.

(iii) Solve for $\frac{dy}{dx}$ (i.e. multiply through by y), and replace y with $f(x)$ to express the derivative explicitly as a function of x.

Example

Find $\frac{dy}{dx}$.

$$y = \frac{x^3(4x-1)^5}{\sqrt[4]{x+5}}$$

Take the log.

$$\ln y = \ln\left(\frac{x^3(4x-1)^5}{\sqrt[4]{x+5}}\right)$$

$$= \ln x^3 + \ln(4x-1)^5 - \ln(x+5)^{\frac{1}{4}}$$

$$= 3\ln x + 5\ln(4x-1) - \frac{1}{4}\ln(x+5)$$

Take the derivative.

$$\frac{d}{dx}\ln y = \frac{d}{dx}\left(3\ln x + 5\ln(4x-1) - \frac{1}{4}\ln(x+5)\right)$$

$$\frac{1}{y}\frac{dy}{dx} = 3\,\frac{1}{x} + 5\,\frac{4}{4x-1} - \frac{1}{4}\,\frac{1}{x+5}$$

$$\frac{dy}{dx} = y\left(\frac{3}{x} + \frac{20}{4x-1} - \frac{1}{4(x+5)}\right)$$

use that $y = \dfrac{x^3(4x-1)^5}{\sqrt[4]{x+5}}$

$$\boxed{\frac{dy}{dx} = \frac{x^3(4x-1)^5}{\sqrt[4]{x+5}}\left(\frac{3}{x} + \frac{20}{4x-1} - \frac{1}{4(x+5)}\right)}$$

Logarithmic Differentiation is required

If $y = x^x$, find $\frac{dy}{dx}$.

Note, the base is variable, so the function is not exponential, and the power if variable, so the function is not a power function. We don't have a rule for this.

We'll use logarithmic differentiation.

$y = x^x \quad , \quad \ln y = \ln x^x = x \ln x$

Use the product rule on the right.

$$\frac{d}{dx} \ln y = \frac{d}{dx} (x \ln x)$$

$$\frac{1}{y}\frac{dy}{dx} = \left(\frac{d}{dx}x\right)\ln x + x\left(\frac{d}{dx}\ln x\right)$$

$$= 1 \cdot \ln x + x \cdot \frac{1}{x}$$

$$= \ln x + 1$$

$$\frac{dy}{dx} = y(\ln x + 1) \qquad \text{but } y = x^x$$

so

$$\frac{d}{dx}x^x = x^x(\ln x + 1)$$

Questions

Find $\frac{dy}{dx}$.

$$y = \frac{(x+3)(x-4)^2}{\sqrt{x}(x+1)^3}$$

$$\ln\left(\frac{(x+3)(x-4)^2}{\sqrt{x}\,(x+1)^3}\right) = \ln\left((x+3)(x-4)^2\right) - \ln\left(\sqrt{x}\,(x+1)^3\right)$$

$$= \ln\left((x+3)(x-4)^2\right) - \left[\ln\sqrt{x} + \ln(x+1)^3\right]$$

(a) $\displaystyle \frac{dy}{dx} = \left[\frac{(x+3)(x-4)^2}{\sqrt{x}(x+1)^3}\right]\left(\frac{1}{x+3} + \frac{2}{x-4} - \frac{1}{2x} - \frac{3}{x+1}\right)$

(b) $\displaystyle \frac{dy}{dx} = \left[\frac{(x+3)(x-4)^2}{\sqrt{x}(x+1)^3}\right]\left(\frac{1}{x+3} + \frac{1}{(x-4)^2} - \frac{1}{\sqrt{x}} - \frac{1}{(x+1)^3}\right)$

(c) $\displaystyle \frac{dy}{dx} = \frac{1}{x+3} + \frac{2}{x-4} - \frac{1}{2x} - \frac{3}{x+1}$

The number e

We have already defined e by the limit

$$\lim_{h \to 0} \frac{e^h - 1}{h} = 1.$$

An alternative definition of the number e is given by

$$\lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n = e$$

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$$

Evaluate the limit

$$\lim_{n\to\infty}\left(1+\frac{2}{n}\right)^n$$

Let k be defined by $\frac{1}{k} = \frac{2}{n}$.

Then $n = 2k$. $n \to \infty$ if and only if $k \to \infty$.

$$\lim_{n\to\infty}\left(1+\frac{2}{n}\right)^n = \lim_{k\to\infty}\left(1+\frac{1}{k}\right)^{2k}$$

Recall $a^{bc} = \left(a^b\right)^c$

$$= \lim_{k\to\infty}\left(\left(1+\frac{1}{k}\right)^k\right)^2$$

$= (e)^2 = e^2$

Recall

if

$\lim_{x \to c} f(x) = L$

then

$\lim_{x \to c} (f(x))^2 = L^2$

Section 4.1: Related Rates

Motivating Example: A spherical balloon is being filled with air. Suppose that we know that the radius is increasing in time at a constant rate of 2 mm/sec. Can we determine the rate at which the surface area of the balloon is increasing at the moment that the radius is 10 cm?

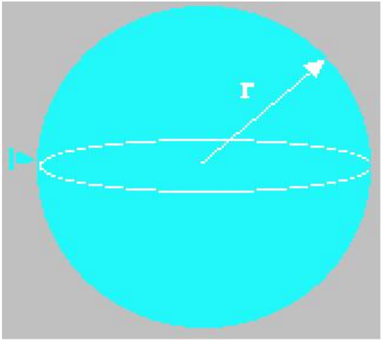

Figure: Spherical Balloon

Example Continued...

Suppose that the radius r and surface area $S = 4\pi r^2$ of a sphere are differentiable functions of time. Write an equation that relates

$$\frac{dS}{dt} \quad \text{to} \quad \frac{dr}{dt}.$$

The chain rule says that $\frac{dS}{dt} = \frac{dS}{dr} \frac{dr}{dt}$

So $\frac{dS}{dt} = \frac{d}{dr}\left(4\pi r^2\right) \cdot \frac{dr}{dt} = 4\pi\left(2r\right)\frac{dr}{dt}$

$$\frac{dS}{dt} = 8\pi r \frac{dr}{dt}$$

Given this result, find the rate at which the surface area is changing when the radius is 10 cm.

we have $\frac{dS}{dt} = 8\pi r \frac{dr}{dt}$ and r is increasing @ 2mm/sec.

rate of change of r i.e. $\frac{dr}{dt}$

$\frac{dr}{dt} = 2 \frac{mm}{sec}$ when $r = 10cm$
$= 100\,mm$

$$\frac{dS}{dt} = 8\pi\,(100\,mm) \cdot 2\,\frac{mm}{sec}$$

$$= 1600\pi\,\frac{mm^2}{sec}$$

The surface area is increasing @ 1600π mm^2 per sec.