Oct 5 Math 2306 sec. 53 Fall 2018

Section 9: Method of Undetermined Coefficients

The context here is linear, constant coefficient, nonhomogeneous equations

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_0 y = g(x)$$

where g comes from the restricted classes of functions

- polynomials,
- exponentials, e^{mx}
- sines and/or cosines, $\sin(kx)$ or $\cos(kx)$
- and products and sums of the above kinds of functions

Recall $y = y_c + y_p$, so we'll have to find both the complementary and the particular solutions!

Examples of Forms of y_p based on g (Trial Guesses)

(a) $g(x) = 1$ (or really any constant) $y_p = A$

(b) $g(x) = x - 7$ $y_p = Ax + B$

(c) $g(x) = 5x^2$ $y_p = Ax^2 + Bx + C$

(d) $g(x) = 3x^3 - 5$ $y_p = Ax^3 + Bx^2 + Cx + D$

(e) $g(x) = xe^{3x}$ $y_p = (Ax + B)e^{3x}$

(f) $g(x) = \cos(7x)$ $y_p = A\cos(7x) + B\sin(7x)$

Examples of Forms of y_p based on g (Trial Guesses)

(g) $g(x) = \sin(2x) - \cos(4x)$

$$y_p = A\sin(2x) + B\cos(2x) + C\sin(4x) + D\cos(4x)$$

(h) $g(x) = x^2\sin(3x)$

Linear combo of 2nd degree polynomials times sine and cosine of 3x

$$y_p = (Ax^2 + Bx + C)\sin(3x) + (Dx^2 + Ex + F)\cos(3x)$$

Examples of Forms of y_p based on g (Trial Guesses)

(i) $g(x) = e^x \cos(2x)$

Linear combo of e^x times sine and cosine of $2x$.

$$y_p = Ae^x \cos(2x) + Be^x \sin(2x)$$

(j) $g(x) = xe^{-x} \sin(\pi x)$

1st degree poly times e^{-x} times sines + cosine of πx.

$$y_p = (Ax+B)e^{-x} \sin(\pi x) + (Cx+D)e^{-x} \cos(\pi x)$$

* Suppose $g(x) = 3x \sin^2(\pi x)$ we'd have to use the ID

$\sin^2\theta = \frac{1}{2} - \frac{1}{2}\cos(2\theta)$ so $g(x) = 3x\left(\frac{1}{2} - \frac{1}{2}\cos(2\pi x)\right) = \frac{3}{2}x - \frac{3}{2}x\cos(2\pi x)$

The Superposition Principle

Example: Determine the correct form of the particular solution using the method of undetermined coefficients for the ODE

$$y'' - 4y' + 4y = 6e^{-3x} + 16x^2$$

By the principle of superposition, if

y_{p_1} solves $y'' - 4y' + 4y = 6e^{-3x}$

and

y_{p_2} solves $y'' - 4y' + 4y = 16x^2$

Then $y_p = y_{p_1} + y_{p_2}$ solves

$$y'' - 4y' + 4y = 6e^{-3x} + 16x^2$$

We know $\quad y_{p_1} = Ae^{-3x}$

and $\quad y_{p_2} = Bx^2 + Cx + D$

so $\quad y_p = Ae^{-3x} + Bx^2 + Cx + D$

On wednesday, we found $A = \frac{6}{25}$

$B = 4$, $C = 8$, $D = 6$ so

$$y_p = \frac{6}{25}e^{-3x} + 4x^2 + 8x + 6$$

A Glitch!

constant coefficient

exponential

$$y'' - y' = 3e^x$$

Suppose

$y_p = Ae^x$

$y_p' = Ae^x$

$y_p'' = Ae^x$

we need $y_p'' - y_p' = 3e^x$

$Ae^x - Ae^x = 3e^x$

$0 = 3e^x$

requiring $0 = 3$

always false.

Let's look @ y_c. y_c solves

$$y'' - y' = 0$$

The characteristic equation is $m^2 - m = 0$

$$m(m-1) = 0$$

$$m = 0 \quad \text{or} \quad m = 1$$ two distinct real roots

$$y_1 = e^{0x} = 1 \qquad y_2 = e^{1x} = e^{x}$$

Our guess for y_p solves the homogeneous equation

$$y_c = c_1 + c_2 e^{x}$$

$$y_p = Ae^{x}$$ Duplicates y_2

We'll fix this by multiplying our "guess" by x (or x to some power)

$$y'' - y' = 3e^x$$

Let's try $y_p = Axe^x$ substitute

$$y_p' = Axe^x + Ae^x$$

$$y_p'' = Axe^x + Ae^x + Ae^x$$

$$= Axe^x + 2Ae^x$$

$$y_p'' - y_p' = Axe^x + 2Ae^x - (Axe^x + Ae^x) = 3e^x$$

Collect like terms e^x and xe^x

$$xe^x(A - A) + e^x(2A - A) = 3e^x$$

$$Ae^x = 3e^x$$

$$A = 3$$

Thus $y_p = 3xe^x$. The general solution

is $y = c_1 + c_2 e^x + 3xe^x$

We'll consider cases

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_0 y = g_1(x) + g_2(x) + \cdots + g_i(x)$$

Using superposition as needed, begin with assumption:

$$y_p = y_{p_1} + \cdots + y_{p_k}$$

where y_{p_i} has the same **general form** as $g_i(x)$.

Case I: y_p as first written has no part that duplicates the complementary solution y_c. Then this first form will suffice.

Case II: y_p has a term y_{p_i} that duplicates a term in the complementary solution y_c. Multiply **that term** (only the y_{p_i} at issue) by x^n, where n is the smallest positive integer that eliminates the duplication.