October 5 Math 2306 sec. 56 Fall 2017

Section 9: Method of Undetermined Coefficients

The context here is linear, constant coefficient, nonhomogeneous equations

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_0 y = g(x)$$

where g comes from the restricted classes of functions

- polynomials,
- exponentials,
- sines and/or cosines,
- and products and sums of the above kinds of functions

We assume that y_p has the same basic form as g.

Examples of Forms of y_p based on g (Trial Guesses)

(a) $g(x) = 1$ (or really any constant), $y_p = A$

(b) $g(x) = x - 7$, $y_p = Ax + B$

(c) $g(x) = 5x^2$, $y_p = Ax^2 + Bx + C$

(d) $g(x) = 3x^3 - 5$, $y_p = Ax^3 + Bx^2 + Cx + D$

(e) $g(x) = xe^{3x}$, $y_p = (Ax + B)e^{3x}$ (f) $g(x) = \cos(7x)$,

$y_p = A\cos(7x) + B\sin(7x)$

(g) $g(x) = \sin(2x) - \cos(4x)$,
$y_p = A\sin(2x) + B\cos(2x) + C\sin(4x) + D\cos(4x)$ (h)

$g(x) = x^2\sin(3x)$,
$y_p = (Ax^2 + Bx + C)\sin(3x) + (Dx^2 + Ex + F)\cos(3x)$

Examples of Forms of y_p based on g (Trial Guesses)

(i) $g(x) = e^x \cos(2x)$

$$y_p = Ae^x \cos(2x) + Be^x \sin(2x)$$

(j) $g(x) = xe^{-x} \sin(\pi x)$

$$y_p = (Ax+B)e^{-x} \sin(\pi x) + (Cx+D)e^{-x} \cos(\pi x)$$

The Superposition Principle

Example: Determine the correct form of the particular solution using the method of undetermined coefficients for the ODE

$$y'' - 4y' + 4y = 6e^{-3x} + 16x^2$$

we can consider the two problems

$$y'' - 4y' + 4y = 6e^{-3x}$$

guess $y_{p_1} = Ae^{-3x}$ we found $A = \frac{6}{25}$

$y_{p_1} = \frac{6}{25}e^{-3x}$ (Sept. 28)

And Consider $y'' - 4y' + 4y = 16x^2$

guess $y_{p_2} = Ax^2 + Bx + C$

we found $A = 4$, $B = 8$, $C = 6$

$$y_{p_2} = 4x^2 + 8x + 6$$

For $y'' - 4y' + 4y = 6e^{-3x} + 16x^2$

$$y_p = y_{p_1} + y_{p_2} = \frac{6}{25}e^{-3x} + 4x^2 + 8x + 6$$

A Glitch!

$$y'' - y' = 3e^x$$

At first glance, it looks like $y_p = Ae^x$.

Substitute

$$y_p' = Ae^x \quad \text{and} \quad y_p'' = Ae^x$$

$$y_p'' - y_p' = Ae^x - Ae^x \stackrel{?}{=} 3e^x$$

$$0e^x = 3e^x$$

this requires $0 = 3$

For undetermined coefficients to work, we'll have to modify our guess.

The associated homogeneous equation is

$$y'' - y' = 0$$

with characteristic eqn $\quad m^2 - m = 0$

$$m(m-1) = 0 \implies m_1 = 0, \; m_2 = 1$$

So $\quad y_c = C_1 e^{0x} + C_2 e^{x} = C_1 + C_2 e^{x}$

* The form of y_p should have been

$$y_p = Axe^{x}$$

We'll consider cases

Using superposition as needed, begin with assumption:

$$y_p = y_{p_1} + \cdots + y_{p_k}$$

where y_{p_i} has the same **general form** as $g_i(x)$.

Case I: y_p as first written has no part that duplicates the complementary solution y_c. Then this first form will suffice.

Case II: y_p has a term y_{p_i} that duplicates a term in the complementary solution y_c. Multiply that term by x^n, where n is the smallest positive integer that eliminates the duplication.

Case II Examples

Solve the ODE

$$y'' - 2y' + y = -4e^x$$

Find y_c: Charac. eqn $m^2 - 2m + 1 = 0$

$(m-1)^2 = 0 \Rightarrow m = 1$ repeated

$y_1 = e^x,\ y_2 = xe^x$

So $y_c = c_1 e^x + c_2 x e^x$

Find y_p: $g(x) =$ constant times e^x

Initial guess $y_p = Ae^x$ matches y_1 won't work

Modify $y_p = Axe^x$ matches y_2 won't work

Modify again $y_p = Ax^2e^x$ This should work!

$$y_p = Ax^2e^x$$

$$y_p' = Ax^2e^x + 2Axe^x$$

$$y_p'' = Ax^2e^x + 2Axe^x + 2Axe^x + 2Ae^x$$

$$= Ax^2e^x + 4Axe^x + 2Ae^x$$

$$y_p'' - 2y_p' + y_p =$$

$$Ax^2e^x + 4Axe^x + 2Ae^x - 2(Ax^2e^x + 2Axe^x) + Ax^2e^x = -4e^x$$

Collect like terms

$$x^2e^x\underbrace{(A-2A+A)}_{=0} + xe^x\underbrace{(4A-4A)}_{=0} + e^x(2A) = -4e^x$$

$$2Ae^x = -4e^x$$

$$A = -2$$

So $\quad y_p = -2x^2e^x$

The general solution to the ODE $y_c + y_p$

$$y = c_1e^x + c_2xe^x - 2x^2e^x$$

Find the form of the particular soluition

$$y'' - 4y' + 4y = \sin(4x) + xe^{2x}$$

Find y_c: $m^2 - 4m + 4 = 0$ $(m-2)^2 = 0$ $\Rightarrow$ $m = 2$ repeated

$y_1 = e^{2x}$, $y_2 = xe^{2x}$ $y_c = c_1 e^{2x} + c_2 x e^{2x}$

To find y_p, we'll consider $y_p = y_{p_1} + y_{p_2}$ where

y_{p_1} solves $y'' - 4y' + 4y = \sin(4x)$ and

y_{p_2} solves $y'' - 4y' + 4y = xe^{2x}$

$y_c = c_1 e^{2x} + c_2 x e^{2x}$

y_{p_1} : $g_1(x) = \sin(4x)$

$$y_{p_1} = A\sin(4x) + B\cos(4x)$$ will work!

y_{p_2} : $g_2(x) = xe^{2x}$

$$y_{p_2} = (Cx+D)e^{2x} = Cxe^{2x} + De^{2x}$$ won't work

$$y_{p_2} = (Cx+D)xe^{2x} = Cx^2e^{2x} + Dxe^{2x}$$ won't work

$$y_{p_2} = (Cx+D)x^2e^{2x} = Cx^3e^{2x} + Dx^2e^{2x}$$ will work

$$y_p = A\sin(4x) + B\cos(4x) + Cx^3e^{2x} + Dx^2e^{2x}$$

Find the form of the particular soluition

$$y''' - y'' + y' - y = \cos x + x^4$$

Find y_c: $m^3 - m^2 + m - 1 = 0$

$(m^3 - m^2) + (m-1) = 0 \Rightarrow m^2(m-1) + (m-1) = 0$

$(m^2+1)(m-1) = 0$

$m^2+1=0 \Rightarrow m^2=-1, \quad m = 0 \pm i \qquad \alpha=0, \ \beta=1$

$y_1 = e^{0x}\cos(1x), \quad y_2 = e^{0x}\sin(1x)$

$m-1=0 \Rightarrow m=1, \quad y_3 = e^x$

$y_c = c_1\cos x + c_2 \sin x + c_3 e^x$

For y_{p_1}, $g_1(x) = \cos x$

$y_{p_1} = A\cos x + B\sin x$ Modify

$y_{p_1} = Ax\cos x + Bx\sin x$ works

For y_{p_2}, $g_2(x) = x^4$

$y_{p_2} = Cx^4 + Dx^3 + Ex^2 + Fx + G$ works

$y_p = Ax\cos x + Bx\sin x + Cx^4 + Dx^3 + Ex^2 + Fx + G$

Find the form of the particular soluition

$$y'' - 2y' + 5y = e^x + 7\sin(2x)$$

Find y_c : $m^2 - 2m + 5 = 0$

$m^2 - 2m + 1 + 4 = 0$

$(m-1)^2 + 4 = 0$

$(m-1)^2 = -4$

$m - 1 = \pm 2i \Rightarrow m = 1 \pm 2i$

$\alpha = 1, \ \beta = 2$

$y_1 = e^x \cos(2x), \quad y_2 = e^x \sin(2x)$

$$y_c = c_1 e^x \cos(2x) + c_2 e^x \sin(2x)$$

For y_{p_1}, $g_1(x) = e^x$

$$y_{p_1} = Ae^x$$ works

For y_{p_2}, $g_2(x) = 7\sin(2x)$

$$y_{p_2} = B\sin(2x) + C\cos(2x)$$ works

$$y_p = Ae^x + B\sin(2x) + C\cos(2x)$$

Solve the IVP

$$y'' - 2y' + y = -4e^x \quad y(0) = -1, \quad y'(0) = 1$$

From several slides back

$$y = c_1 e^x + c_2 x e^x - 2x^2 e^x$$

$$y' = c_1 e^x + c_2 x e^x + c_2 e^x - 2x^2 e^x - 4x e^x$$

$$y(0) = c_1 e^0 + c_2 \cdot 0 e^0 - 2 \cdot 0^2 e^0 = c_1 = -1$$

$$y'(0) = c_1 e^0 + c_2 \cdot 0 e^0 + c_2 e^0 - 2 \cdot 0^2 e^0 - 4 \cdot 0 e^0 = 1$$

$$c_1 + c_2 = 1$$

$C_1 = -1$ so $C_2 = 1 - C_1 = 1 - (-1) = 2$

The solution to the IVP is

$$y = -e^x + 2xe^x - 2x^2e^x$$

Section 10: Variation of Parameters

We are still considering nonhomogeneous, linear ODEs. Consider equations of the form

$$y'' + y = \tan x, \quad \text{or} \quad x^2y'' + xy' - 4y = e^x.$$

The method of undetermined coefficients is not applicable to either of these. We require another approach.

Variation of Parameters

For the equation in standard form

$$\frac{d^2y}{dx^2}+P(x)\frac{dy}{dx}+Q(x)y=g(x),$$

suppose $\{y_1(x),\ y_2(x)\}$ is a fundamental solution set for the associated homogeneous equation. We seek a particular solution of the form

$$y_p(x)=u_1(x)y_1(x)+u_2(x)y_2(x)$$

where u_1 and u_2 are functions we will determine (in terms of y_1, y_2 and g).

This method is called **variation of parameters**.