October 5 Math 3260 sec. 57 Fall 2017

Section 4.3: Linearly Independent Sets and Bases

We defined **linear dependence/independence** and **span** for elements of general vector spaces. Then, we defined a basis.

Definition: Let H be a subspace of a vector space V. An indexed set of vectors $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_p\}$ in V is a **basis** of H provided

(i) $\mathcal{B}$ is linearly independent, and

(ii) $H =$Span$(\mathcal{B})$.

We can think of a basis as a *minimal spanning set*. All of the *information* needed to construct vectors in H is contained in the basis, and none of this information is repeated.

A Spanning Set Theorem

Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ be a set in a vector space V and $H = $Span$(S)$.

(a.) If one of the vectors in S, say $\mathbf{v}_k$ is a linear combination of the other vectors in S, then the subset of S obtained by eliminating $\mathbf{v}_k$ still spans H.

(b) If $H \neq \{\mathbf{0}\}$, then some subset of S is a basis for H.

If we start with a spanning set, we can eliminate *duplication* and arrive at a basis.

Column Space

Find a basis for the column space matrix B that is in reduced row echelon form

$$B = \begin{array}{c} \begin{bmatrix} 1 & 4 & 0 & 2 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\ \begin{array}{ccccc} \vec{b}_1 & \vec{b}_2 & \vec{b}_3 & \vec{b}_4 & \vec{b}_5 \end{array} \end{array}.$$

$\vec{b}_2 = 4\vec{b}_1$

$\vec{b}_4 = 2\vec{b}_1 - \vec{b}_3$

$\{\vec{b}_1, \vec{b}_3, \vec{b}_5\}$ is linearly independent

So this is a basis for Col B.

Using the rref

Theorem: If $A = [\mathbf{a}_1 \cdots \mathbf{a}_n]$ and $B = [\mathbf{b}_1 \cdots \mathbf{b}_n]$ are row equivalent matrices, then Nul $A =$ Nul B. That is, the equations

$$A\mathbf{x} = \mathbf{0} \quad \text{and} \quad B\mathbf{x} = \mathbf{0}$$

have the same solution set.

Note what this means! It means that $\{\mathbf{a}_1, \dots, \mathbf{a}_n\}$ and $\{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ have **exactly the same linear dependence relationships**!

Theorem:

The pivot columns of a matrix A form a basis of Col A.

Caveat: This means we can use row reduction to identify a basis, but the vectors we obtain will be from the original matrix A. (As illustrated in the following example.)

Find a basis for Col A[1]

$$A = \begin{bmatrix} 1 & 4 & 0 & 2 & -1 \\ 3 & 12 & 1 & 5 & 5 \\ 2 & 8 & 1 & 3 & 2 \\ 5 & 20 & 2 & 8 & 8 \end{bmatrix}.$$

we'll use rref of A

$\downarrow$ rref

$$\begin{bmatrix} 1 & 4 & 0 & 2 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$\uparrow \quad \uparrow \quad \uparrow$

pivot columns

The pivot columns of A form a basis for Col A

$$\text{Col}A = \text{Span}\left\{ \begin{bmatrix} 1 \\ 3 \\ 2 \\ 5 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 5 \\ 2 \\ 8 \end{bmatrix} \right\}$$

[1] Use a calculator to do the row reduction.

Find bases for Nul A and Col A

$$A = \begin{bmatrix} 1 & 0 & 3 & -2 \\ 2 & 1 & 5 & 1 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & 3 & -2 \\ 0 & 1 & -1 & 5 \end{bmatrix}$$

pivot columns

$$\text{Col } A = \text{span}\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$$

From the rref, $A\vec{x} = \vec{0}$ requires

$$x_1 = -3x_3 + 2x_4$$

$$x_2 = x_3 - 5x_4$$

x_3, x_4 - free

So
$$\vec{x} = \begin{bmatrix} -3x_3+2x_4 \\ x_3-5x_4 \\ x_3 \\ x_4 \end{bmatrix} = x_3\begin{bmatrix} -3 \\ 1 \\ 1 \\ 0 \end{bmatrix} + x_4\begin{bmatrix} 2 \\ -5 \\ 0 \\ 1 \end{bmatrix}$$

So
$$\text{Nul } A = \text{Span}\left\{ \begin{bmatrix} -3 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -5 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Both spanning sets are bases for the respective vector space.

Section 4.4: Coordinate Systems

We begin with a theorem about uniqueness of linear combinations (of linearly independent vectors).

Theorem: Let $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ be a basis for a vector space V. Then for each vector $\mathbf{x}$ in V, there is a unique set of scalars $c_1, \ldots, c_n$ such that

$$\mathbf{x} = c_1\mathbf{b}_1 + \cdots c_n\mathbf{b}_n.$$

The key point here is that there is exactly one set of coefficients (c_i) for a given $\vec{x}$.

Suppose $\quad \vec{x} = c_1\vec{b}_1 + c_2\vec{b}_2 + \cdots + c_n\vec{b}_n$

and $\quad \vec{x} = d_1\vec{b}_1 + d_2\vec{b}_2 + \cdots + d_n\vec{b}_n$

$\vec{x} - \vec{x} = \vec{0}$ so

$$\vec{0} = (c_1 - d_1)\vec{b}_1 + (c_2 - d_2)\vec{b}_2 + \dots + (c_n - d_n)\vec{b}_n$$

Since the $\vec{b}$'s are linearly independent, each coefficient must be zero.

That is
$$\left.\begin{array}{c} c_1 - d_1 = 0 \\ c_2 - d_2 = 0 \\ \vdots \\ \text{etc} \end{array}\right\} \Rightarrow \quad c_i = d_i \text{ for } i = 1, \dots, n.$$

Coordinate Vectors

Definition: Let $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ be an **ordered** basis of the vector space V. For each $\mathbf{x}$ in V we define the **coordinate vector of x relative to the basis** $\mathcal{B}$ to be the unique vector $(c_1, \ldots, c_n)$ in $\mathbb{R}^n$ where these entries are the weights $\mathbf{x} = c_1\mathbf{b}_1 + \cdots c_n\mathbf{b}_n$.

We'll use the notation

$$\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = [\mathbf{x}]_{\mathcal{B}}.$$

Example

Let $\mathcal{B} = \{1, t, t^2, t^3\}$ (in that order) in $\mathbb{P}_3$. Determine $[\mathbf{p}]_{\mathcal{B}}$ where

(a) $\mathbf{p}(t) = 3 - 4t^2 + 6t^3$ $= 3 \cdot 1 + 0 \cdot t + (-4)t^2 + 6t^3$

$$[\vec{p}]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 0 \\ -4 \\ 6 \end{bmatrix}$$

(b) $\mathbf{p}(t) = p_0 + p_1 t + p_2 t^2 + p_3 t^3$

$$[\vec{p}]_{\mathcal{B}} = \begin{bmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \end{bmatrix}$$

Example

Let $\mathbf{b}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $\mathbf{b}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, and $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$. Find $[\mathbf{x}]_{\mathcal{B}}$ for $\mathbf{x} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$.

$$[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \quad \text{where}$$

$$c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$$

This equation is equivalent to

$$\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$$

Using row reduction

$$\begin{bmatrix} 2 & -1 & 4 \\ 1 & 1 & 5 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \end{bmatrix}$$

$$[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

$$\left[\begin{bmatrix} 4 \\ 5 \end{bmatrix}\right]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

Coordinates in $\mathbb{R}^n$

Note from this example that $\mathbf{x} = P_{\mathcal{B}}[\mathbf{x}]_{\mathcal{B}}$ where $P_{\mathcal{B}}$ is the matrix $[\mathbf{b}_1\ \mathbf{b}_2]$. The matrix $P_{\mathcal{B}}$ is called the **change of coordinates matrix** for the basis $\mathcal{B}$ (or from the basis $\mathcal{B}$ to the standard basis).

Let $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ be an ordered basis of $\mathbb{R}^n$. Then the change of coordinate mapping $\mathbf{x} \mapsto [\mathbf{x}]_{\mathcal{B}}$ is the linear transformation defined by

$$[\mathbf{x}]_{\mathcal{B}} = P_{\mathcal{B}}^{-1}\mathbf{x}$$

where the matrix

$$P_{\mathcal{B}} = [\mathbf{b}_1 \quad \mathbf{b}_2 \quad \cdots \quad \mathbf{b}_n].$$

Example

Let $\mathcal{B} = \left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$. Determine the matrix $P_{\mathcal{B}}$ and its inverse.

$$P_{\mathcal{B}} = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} \qquad \det(P_{\mathcal{B}}) = 2+1 = 3 \qquad P_{\mathcal{B}}^{-1} = \frac{1}{3}\begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}$$

Use this to find

(a) the coordinate vector of $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$$\left[\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right]_{\mathcal{B}} = P_{\mathcal{B}}^{-1}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

(b) the coordinate vector of $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$$\left[\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right]_{\mathcal{B}} = P_{\mathcal{B}}^{-1} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

(c) a vector $\mathbf{x}$ whose coordinate vector is $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

$$\vec{x} = P_{\mathcal{B}} [\vec{x}]_{\mathcal{B}} = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

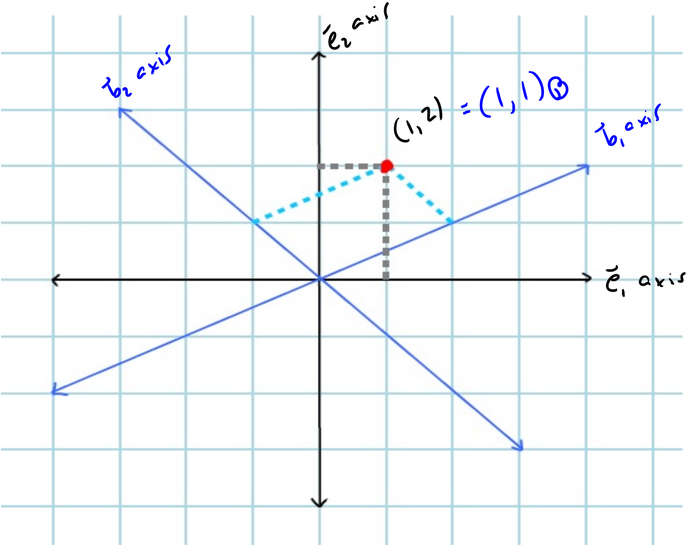

Figure: $\mathbb{R}^2$ shown using elementary basis $\{(1,0),(0,1)\}$ and with the alternative basis $\{(2,1),(-1,1)\}$.