# October 5 Math 3260 sec. 58 Fall 2017

## Section 4.3: Linearly Independent Sets and Bases

We defined **linear dependence/independence** and **span** for elements of general vector spaces. Then, we defined a basis.

**Definition:** Let $H$ be a subspace of a vector space $V$. An indexed set of vectors $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_p\}$ in $V$ is a **basis** of $H$ provided

(i) $\mathcal{B}$ is linearly independent, and

(ii) $H =$Span$(\mathcal{B})$.

We can think of a basis as a *minimal spanning set*. All of the *information* needed to construct vectors in $H$ is contained in the basis, and none of this information is repeated.

# A Spanning Set Theorem

Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ be a set in a vector space $V$ and $H =$Span$(S)$.

(a.) If one of the vectors in $S$, say $\mathbf{v}_k$ is a linear combination of the other vectors in $S$, then the subset of $S$ obtained by eliminating $\mathbf{v}_k$ still spans $H$.

(b) If $H \neq \{\mathbf{0}\}$, then some subset of $S$ is a basis for $H$.

If we start with a spanning set, we can eliminate *duplication* and arrive at a basis.

# Using the rref

**Theorem:** If $A = [\mathbf{a}_1 \cdots \mathbf{a}_n]$ and $B = [\mathbf{b}_1 \cdots \mathbf{b}_n]$ are row equivalent matrices, then Nul $A$ = Nul $B$. That is, the equations

$$A\mathbf{x} = \mathbf{0} \quad \text{and} \quad B\mathbf{x} = \mathbf{0}$$

have the same solution set.

Note what this means! It means that $\{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$ and $\{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ have **exactly the same linear dependence relationships**!

# Theorem:

**The pivot columns of a matrix $A$ form a basis of Col $A$.**

Caveat: This means we can use row reduction to identify a basis, but the vectors we obtain will be from the original matrix $A$. (As illustrated in the following example.)

# Find a basis for Col $A$[1]

$$A = \begin{bmatrix} 1 & 4 & 0 & 2 & -1 \\ 3 & 12 & 1 & 5 & 5 \\ 2 & 8 & 1 & 3 & 2 \\ 5 & 20 & 2 & 8 & 8 \end{bmatrix}. \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 4 & 0 & 2 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{array}{ccccc} \uparrow & & \uparrow & & \uparrow \end{array}$$

pivot columns

A basis for Col$A$ is $\left\{ \begin{bmatrix} 1 \\ 3 \\ 2 \\ 5 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 5 \\ 2 \\ 8 \end{bmatrix} \right\}$

[1]Use a calculator to do the row reduction.

# Find bases for Nul $A$ and Col $A$

$$A = \begin{bmatrix} 1 & 0 & 3 & -2 \\ 2 & 1 & 5 & 1 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & 3 & -2 \\ 0 & 1 & -1 & 5 \end{bmatrix}$$

$\uparrow \quad \uparrow$
pivot columns

A basis for Col $A$ is $\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$

If $\vec{x}$ solves $A\vec{x} = \vec{0}$ then

$$x_1 = -3x_3 + 2x_4$$
$$x_2 = x_3 - 5x_4$$
$x_3, x_4$ - free

$$\vec{x} = \begin{bmatrix} -3x_3 + 2x_4 \\ x_3 - 5x_4 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} -3 \\ 1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 2 \\ -5 \\ 0 \\ 1 \end{bmatrix}$$

A basis for Nul $A$ is $\left\{ \begin{bmatrix} -3 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -5 \\ 0 \\ 1 \end{bmatrix} \right\}$

# Section 4.4: Coordinate Systems

We begin with a theorem about uniqueness of linear combinations (of linearly independent vectors).

**Theorem:** Let $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ be a basis for a vector space $V$. Then for each vector $\mathbf{x}$ in $V$, there is a unique set of scalars $c_1, \ldots, c_n$ such that

$$\mathbf{x} = c_1\mathbf{b}_1 + \cdots c_n\mathbf{b}_n.$$

The point of this theorem is that there is only one set of coefficients for a given $\vec{x}$.

Suppose $\vec{x} = c_1\vec{b}_1 + c_2\vec{b}_2 + \cdots + c_n\vec{b}_n$

and $\vec{x} = a_1\vec{b}_1 + a_2\vec{b}_2 + \cdots + a_n\vec{b}_n$

$$\vec{x} - \vec{x} = \vec{0}$$

$$\vec{0} = (c_1 - a_1)\vec{b}_1 + (c_2 - a_2)\vec{b}_2 + \dots + (c_n - a_n)\vec{b}_n$$

Since the $\vec{b}$'s are linearly independent,

$$\left.\begin{array}{l} c_1 - a_1 = 0 \\ c_2 - a_2 = 0 \\ \vdots \\ c_n - a_n = 0 \end{array}\right\} \Rightarrow \quad c_i = a_i \quad \text{for } i = 1, \dots, n.$$

# Coordinate Vectors

**Definition:** Let $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ be an **ordered** basis of the vector space $V$. For each $\mathbf{x}$ in $V$ we define the **coordinate vector of x relative to the basis** $\mathcal{B}$ to be the unique vector $(c_1, \ldots, c_n)$ in $\mathbb{R}^n$ where these entries are the weights $\mathbf{x} = c_1\mathbf{b}_1 + \cdots c_n\mathbf{b}_n$.

We'll use the notation

$$\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = [\mathbf{x}]_{\mathcal{B}}.$$

# Example

Let $\mathcal{B} = \{1, t, t^2, t^3\}$ (in that order) in $\mathbb{P}_3$. Determine $[\mathbf{p}]_{\mathcal{B}}$ where

(a) $\mathbf{p}(t) = 3 - 4t^2 + 6t^3$ $= 3\cdot 1 + 0\cdot t + (-4)t^2 + 6t^3$

$$[\vec{p}]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 0 \\ -4 \\ 6 \end{bmatrix}$$

(b) $\mathbf{p}(t) = p_0 + p_1 t + p_2 t^2 + p_3 t^3$

$$[\vec{p}]_{\mathcal{B}} = \begin{bmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \end{bmatrix}$$

# Example

Let $\mathbf{b}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $\mathbf{b}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, and $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$. Find $[\mathbf{x}]_{\mathcal{B}}$ for $\mathbf{x} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$.

$[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$ such that

$$c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$$

we can write this as the matrix equation

$$\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$$

The matrix is $[\vec{b}_1 \ \vec{b}_2]$. Using row reduction

$$\begin{bmatrix} 2 & -1 & 4 \\ 1 & 1 & 5 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \end{bmatrix}$$

so

$$\left[\begin{bmatrix} 4 \\ 5 \end{bmatrix}\right]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

# Coordinates in $\mathbb{R}^n$

Note from this example that $\mathbf{x} = P_{\mathcal{B}}[\mathbf{x}]_{\mathcal{B}}$ where $P_{\mathcal{B}}$ is the matrix $[\mathbf{b}_1\ \mathbf{b}_2]$. The matrix $P_{\mathcal{B}}$ is called the **change of coordinates matrix** for the basis $\mathcal{B}$ (or from the basis $\mathcal{B}$ to the standard basis).

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Let $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ be an ordered basis of $\mathbb{R}^n$. Then the change of coordinate mapping $\mathbf{x} \mapsto [\mathbf{x}]_{\mathcal{B}}$ is the linear transformation defined by

$$[\mathbf{x}]_{\mathcal{B}} = P_{\mathcal{B}}^{-1}\mathbf{x}$$

where the matrix

$$P_{\mathcal{B}} = [\mathbf{b}_1 \quad \mathbf{b}_2 \quad \cdots \quad \mathbf{b}_n].$$

## Example

Let $\mathcal{B} = \left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$. Determine the matrix $P_{\mathcal{B}}$ and its inverse.

$$P_{\mathcal{B}} = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}, \quad \det(P_{\mathcal{B}}) = 3 \qquad P_{\mathcal{B}}^{-1} = \frac{1}{3}\begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}$$

Use this to find

(a) the coordinate vector of $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$$\left[\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right]_{\mathcal{B}} = P_{\mathcal{B}}^{-1}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

(b) the coordinate vector of $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$$\left[\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right]_{\mathcal{B}} = P_{\mathcal{B}}^{-1}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

(c) a vector **x** whose coordinate vector is $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

$$\vec{x} = P_{\mathcal{B}}[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

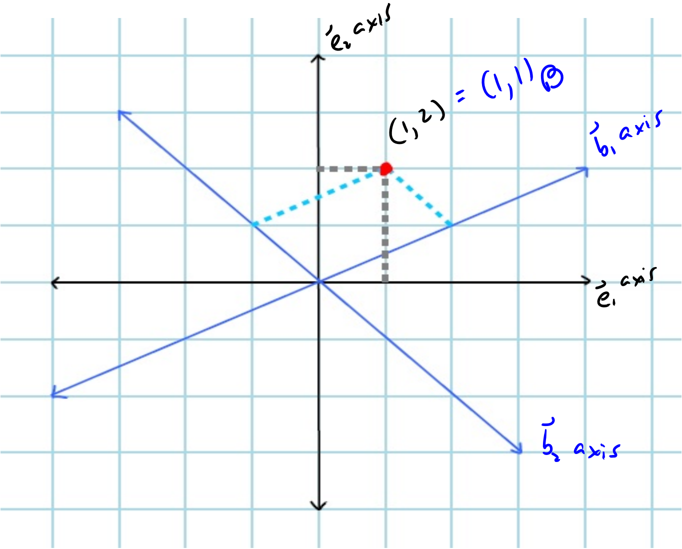


Figure: $\mathbb{R}^2$ shown using elementary basis $\{(1,0),(0,1)\}$ and with the alternative basis $\{(2,1),(-1,1)\}$.