

Section 4.6: Variation of Parameters

Consider the equation $y'' + y = \tan x$. What happens if we try to find a particular solution **having the same form** as the right hand side?

Perhaps $y_p = A \tan x$

$$y_p' = A \sec^2 x \quad \leftarrow \text{might have to match } \sec^2 x$$

Perhaps $y_p = A \tan x + B \sec^2 x$

$$y_p' = A \sec^2 x + 2B \sec x \sec x \tan x \quad \leftarrow \begin{matrix} \text{might} \\ \text{need} \\ \sec^2 x \tan x \end{matrix}$$

perhaps

$$y_p = A \tan x + B \sec^2 x + C \sec^2 x \tan x$$

$$y_p' = A \sec^2 x + 2B \sec^2 x \tan x + C \sec^4 x + 2C \sec^2 x \tan^2 x$$

Each derivative gives a
new term ! this
won't work!

We need another
method !

Consider the equation $x^2y'' + xy' - 4y = e^x$. What happens if we assume $y_p = Ae^x$?

Suppose $y_p = Ae^x$, $y_p' = Ae^x$, $y_p'' = Ae^x$

$$x^2 y_p'' + x y_p' - 4 y_p = e^x$$

$$x^2 A e^x + x A e^x - 4 A e^x = e^x$$

$$\underline{\underline{A}} \underline{\underline{x^2 e^x}} + \underline{\underline{A x e^x}} - \underline{\underline{4 A e^x}} = \underline{\underline{e^x}} + \underline{\underline{0 x e^x}} + \underline{\underline{0 x^2 e^x}}$$

$$A = 0 \quad -4A = 1$$

$$A = 0$$

This requires
 $A = 0 = \frac{1}{4}$
which is impossible!

Again, we need a new
method.

We need another method!

For the equation in standard form

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = g(x),$$

suppose $\{y_1(x), y_2(x)\}$ is a fundamental solution set for the associated homogeneous equation. We seek a particular solution of the form

$$y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x)$$

where u_1 and u_2 are functions we will determine (in terms of y_1 , y_2 and g).

This method is called **variation of parameters**.

Variation of Parameters: Derivation of y_p

$$y'' + P(x)y' + Q(x)y = g(x)$$

Set $y_p = u_1(x)y_1(x) + u_2(x)y_2(x)$

$$y_p' = u_1 y_1' + u_2 y_2' + \underbrace{u_1' y_1 + u_2' y_2}_{0}$$

Let's assume

$$u_1' y_1 + u_2' y_2 = 0$$

$$y_p'' = u_1 y_1'' + u_2 y_2'' + u_1' y_1' + u_2' y_2'$$

Remember that $y_i'' + P(x)y_i' + Q(x)y_i = 0$, for $i = 1, 2$

$$y_p'' + P(x)y_p' + Q(x)y_p = g(x)$$

$$u_1 y_1'' + u_2 y_2'' + u_1' y_1' + u_2' y_2' + P(x)(u_1 y_1' + u_2 y_2') +$$

$$+ Q(x)(u_1 y_1 + u_2 y_2) = g(x)$$

Collect by u_1, u_2, u_1', u_2'

$$\underbrace{u_1(y_1'' + P(x)y_1' + Q(x)y_1)}_{o''} + \underbrace{u_2(y_2'' + P(x)y_2' + Q(x)y_2)}_{o''} + u_1' y_1' + u_2' y_2' = g(x)$$

We have two equations for u_1 and u_2

$$y_1 u_1' + y_2 u_2' = 0$$

and

$$y_1' u_1' + y_2' u_2' = g(x)$$

$$\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} \begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} 0 \\ g(x) \end{pmatrix}$$

Use Cramer's Rule:

$$W_1 = \begin{vmatrix} 0 & y_2 \\ g(x) & y_2' \end{vmatrix}, \quad W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & g(x) \end{vmatrix}$$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = W(y_1, y_2)(x) \quad \text{The Wronskian of } y_1, y_2$$

$$u_1' = \frac{w_1}{W} = -\frac{g(x)y_2}{W} \Rightarrow u_1 = \int -\frac{y_2 g(x)}{W} dx$$

$$u_2' = \frac{w_2}{W} = \frac{g(x)y_1}{W} \Rightarrow u_2 = \int \frac{y_1 g(x)}{W} dx$$

Example:

Solve the ODE $y'' + y = \tan x$.

Find y_1, y_2 : $y'' + y = 0$ $m^2 + 1 = 0 \Rightarrow m = \pm i$
 $\alpha = 0, \beta = 1$

$$y_1 = \cos x, \quad y_2 = \sin x$$

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1$$

$$g(x) = \tan x$$

$$u_1 = \int -\frac{y_2 g}{W} dx = \int -\frac{\sin x \tan x}{1} dx$$

$$u_2 = \int \frac{y_1 g}{w} dx = \int \frac{\cos x \tan x}{1} dx = \int \sin x dx$$

Integrate :

$$\begin{aligned} u_1 &= \int -\sin x \tan x dx = \int \frac{-\sin^2 x}{\cos x} dx = \int -\frac{(1-\cos^2 x)}{\cos x} dx \\ &= \int (-\sec x + \cos x) dx = -\ln|\sec x + \tan x| + \sin x \end{aligned}$$

$$u_2 = \int \sin x dx = -\cos x$$

$$y_p = u_1 y_1 + u_2 y_2$$

$$= (-\ln|\sec x + \tan x| + \sin x) \cos x + (-\cos x) \sin x$$

$$= -\cos x \ln|\sec x + \tan x| + \sin x \cos x - \cos x \sin x$$

$$= -\cos x \ln|\sec x + \tan x|$$

The general solution to the DE is

$$y = C_1 \cos x + C_2 \sin x - \cos x \ln|\sec x + \tan x|$$

Example:

Solve the ODE

$$y'' - 2y' + y = \frac{e^x}{1+x^2}.$$

Find y_1, y_2 : $y'' - 2y' + y = 0$

$$\begin{aligned}m^2 - 2m + 1 &= 0 \\(m-1)^2 &= 0 \Rightarrow m = 1 \text{ repeated}\end{aligned}$$

$$y_1 = e^x, \quad y_2 = xe^x$$

$$W(y_1, y_2) = \begin{vmatrix} e^x & xe^x \\ e^x & e^x + xe^x \end{vmatrix} = e^{2x} + xe^{2x} - xe^{2x} = e^{2x}$$

$$g(x) = \frac{e^x}{1+x^2}$$

$$u_1 = \int -\frac{y_2 g_2}{w} dx = \int \frac{-x e^x \left(\frac{e^x}{1+x^2} \right)}{e^{2x}} dx = \int \frac{-x e^{2x}}{e^{2x}(1+x^2)} dx$$

$$= - \int \frac{x}{1+x^2} dx = -\frac{1}{2} \operatorname{Dn}(1+x^2)$$

$$u_2 = \int \frac{y_1 g_2}{w} dx = \int \frac{e^x \left(\frac{e^x}{1+x^2} \right)}{e^{2x}} dx = \int \frac{e^{2x}}{e^{2x}(1+x^2)} dx$$

$$= \int \frac{1}{1+x^2} dx = \operatorname{tan}^{-1} x$$

$$y_p = u_1 y_1 + u_2 y_2$$

$$= -\frac{1}{2} \ln(1+x^2) e^x + \tan^{-1} x (x e^x)$$

The general solution is

$$y = C_1 e^x + C_2 x e^x - \frac{x}{2} \ln(1+x^2) + x e^x \tan^{-1} x .$$