October 7 Math 2306 sec 54 Fall 2015

Section 4.6: Variation of Parameters

Consider the equation $y'' + y = \tan x$. What happens if we try to find a particular solution having the same *form* as the right hand side?

we could try $y_p = A\tan x$

$y_p' = A\sec^2 x$ ← might need to match $\sec^2 x$

perhaps $y_p = A\tan x + B\sec^2 x$

$y_p' = A\sec^2 x + 2B\sec^2 x\tan x$ ← now we have $\sec^2 x\tan x$

try $$y_p = A\tan x + B\sec^2 x + C\sec^2 x\tan x$$

$$y_p' = A\sec^2 x + 2B\sec^2 x\tan x + C\sec^4 x + 2C\sec^2 x\tan^2 x$$

New functions keep appearing

this method won't work!

Consider the equation $x^2y'' + xy' - 4y = e^x$. What happens if we assume $y_p = Ae^x$?

$y_p' = Ae^x \quad , \quad y_p'' = Ae^x$

$x^2 y_p'' + x\, y_p' - 4y_p = e^x$

$x^2Ae^x + xAe^x - 4Ae^x = e^x + 0x^2e^x + 0xe^x$

$A = 0 \qquad -4A = 1$

$A = 0$

This requires $A = 0 = \frac{-1}{4}$ which isn't possible!

We need another method!

For the equation in standard form

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x) = g(x),$$

suppose $\{y_1(x),\ y_2(x)\}$ is a fundamental solution set for the associated homogeneous equation. We seek a particular solution of the form

$$y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x)$$

where u_1 and u_2 are functions we will determine (in terms of y_1, y_2 and g).

This method is called **variation of parameters**.

Variation of Parameters: Derivation of y_p

$$y'' + P(x)y' + Q(x)y = g(x)$$

Set $\quad y_p = u_1(x)y_1(x) + u_2(x)y_2(x)$

$$y_p' = u_1 y_1' + u_2 y_2' + \underbrace{u_1' y_1 + u_2' y_2}_{\overset{\text{set}}{=} 0}$$

Let's assume

$u_1' y_1 + u_2' y_2 = 0$

$$y_p'' = u_1' y_1' + u_2' y_2' + u_1 y_1'' + u_2 y_2''$$

Remember that $\quad y_i'' + P(x)y_i' + Q(x)y_i = 0, \quad$ for $i = 1, 2$

$$y_p'' + P(x)\,y_p' + Q(x)\,y_p = g(x)$$

$$u_1' y_1' + u_2' y_2' + u_1 y_1'' + u_2 y_2'' + P(x)\left(u_1 y_1' + u_2 y_2'\right)$$

$$+ Q(x)\left(u_1 y_1 + u_2 y_2\right) = g(x)$$

$$u_1' y_1' + u_2' y_2' + u_1\left(\underbrace{y_1'' + P(x) y_1' + Q(x) y_1}_{0''}\right)$$

$$+ u_2\left(\underbrace{y_2'' + P(x) y_2' + Q(x) y_2}_{0''}\right) = g(x)$$

We have 2 equations for our 2 unknowns

$$y_1 u_1' + y_2 u_2' = 0$$

$$y_1' u_1' + y_2' u_2' = g(x)$$

$$\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} \begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} 0 \\ g(x) \end{pmatrix}$$

Set $W_1 = \begin{vmatrix} 0 & y_2 \\ g & y_2' \end{vmatrix}$, $W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & g \end{vmatrix}$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

This is the Wronskian $W(y_1, y_2)$

$$u_1' = \frac{W_1}{W} = \frac{-g(x)\, y_2}{W}$$

$$u_2' = \frac{W_2}{W} = \frac{g(x)\, y_1}{W}$$

$$u_1 = \int \frac{-y_2(x)\, g(x)}{W(x)}\, dx$$

$$u_2 = \int \frac{y_1(x)\, g(x)}{W(x)}\, dx$$

Example:

Solve the ODE $y'' + y = \tan x$.

$g(x) = \tan x$

Find y_1, y_2 : $\quad y'' + y = 0 \qquad m^2 + 1 = 0 \Rightarrow m = \pm i$

$\alpha = 0, \ \beta = 1$

$y_1 = \cos x, \quad y_2 = \sin x$

$$W(y_1, y_2) = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1$$

$$u_1 = \int \frac{-y_2 g}{W}\, dx = \int -\frac{\sin x \tan x}{1}\, dx$$

$$u_1 = \int -\sin x \tan x \, dx = \int \frac{-\sin^2 x}{\cos x} \, dx$$

$$= -\int \frac{(1-\cos^2 x)}{\cos x} \, dx = \int \left(-\sec x + \cos x\right) dx$$

$$= -\ln\left|\sec x + \tan x\right| + \sin x$$

$$u_2 = \int \frac{y_1 g}{W} \, dx = \int \frac{\cos x \tan x}{1} \, dx = \int \sin x \, dx$$

$$= -\cos x$$

$$y_p = u_1 y_1 + u_2 y_2$$

$$= \left(-\ln|\sec x + \tan x| + \sin x\right)\cos x + (-\cos x)\sin x$$

$$= -\cos x \ln|\sec x + \tan x| + \sin x \cos x - \cos x \sin x$$

$$y_p = -\cos x \ln|\sec x + \tan x|$$

The general solution to the DE is

$$y = C_1 \cos x + C_2 \sin x - \cos x \ln|\sec x + \tan x|$$

Example:

Solve the ODE

$$y'' - 2y' + y = \frac{e^x}{1+x^2}.$$

$g(x) = \dfrac{e^x}{1+x^2}$

Find y_1, y_2: $\quad y'' - 2y' + y = 0 \qquad m^2 - 2m + 1 = 0$

$(m-1)^2 = 0 \Rightarrow m = 1$ repeated

$y_1 = e^x \quad , \quad y_2 = xe^x$

$$W = \begin{vmatrix} e^x & xe^x \\ e^x & e^x + xe^x \end{vmatrix} = e^{2x} + xe^{2x} - xe^{2x} = e^{2x}$$

$$u_1 = \int \frac{-y_2 g}{W}\,dx = \int \frac{-xe^x\left(\frac{e^x}{1+x^2}\right)}{e^{2x}}\,dx$$

$$= \int \frac{-xe^{2x}}{e^{2x}(1+x^2)}\,dx = \int \frac{-x}{1+x^2}\,dx = \frac{-1}{2}\ln(1+x^2)$$

$$u_2 = \int \frac{y_1 g}{W}\,dx = \int \frac{e^x\left(\frac{e^x}{1+x^2}\right)}{e^{2x}}\,dx$$

$$= \int \frac{e^{2x}}{e^{2x}(1+x^2)}\,dx = \int \frac{1}{1+x^2}\,dx = \tan^{-1} x$$

$$y_p = u_1 y_1 + u_2 y_2$$

$$= \frac{-1}{2}\ln(1+x^2)\,e^x + \tan^{-1}x\,(xe^x)$$

The general solution to the ODE is

$$\boxed{y = C_1 e^x + C_2 x e^x - \frac{e^x}{2}\ln(1+x^2) + xe^x \tan^{-1}x.}$$