October 8 MATH 1113 sec. 52 Fall 2018

Section 5.2: Exponential Functions

Many models involve quantities that change at a rate proportional to the quantity itself.

- money in an account with compounded interest,
- bacteria growing in a culture,
- the mass of a substance undergoing radio active decay

These grow or decay in a fashion that is very different from polynomials and rational functions.

The study of how functions change is a big part of Calculus. Here, we will define **exponential** functions and examine some of their properties.

Exponential Functions

Definition: Let a be a positive real number different from 1—i.e. $a > 0$ and $a \neq 1$. The function

$$f(x) = a^x$$

is called the **exponential function of base** a. Its domain is $(-\infty, \infty)$, and its range is $(0, \infty)$.

Some examples of exponential functions are

$$f(x) = 2^x, \qquad g(x) = \left(\frac{1}{3}\right)^x, \quad \text{and} \quad h(x) = \pi^{x-1} \;=\; \pi^{-1}\pi^{x}$$

Observation

We don't want to confuse exponential and power functions. Note that in an **exponential** function

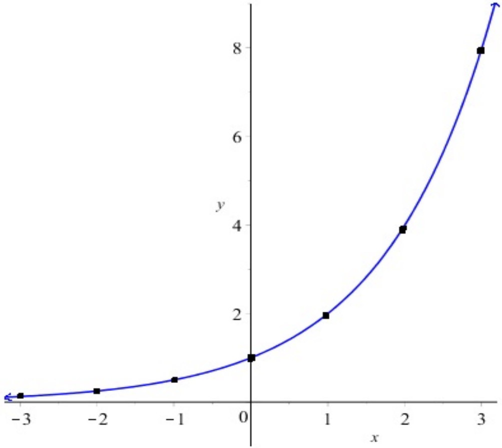
$$f(x) = 4^x$$

the base is a constant, and the exponent is a variable. Contrast a **power** function

$$f(x) = x^4$$

in which the base is variable, and the exponent is a constant.

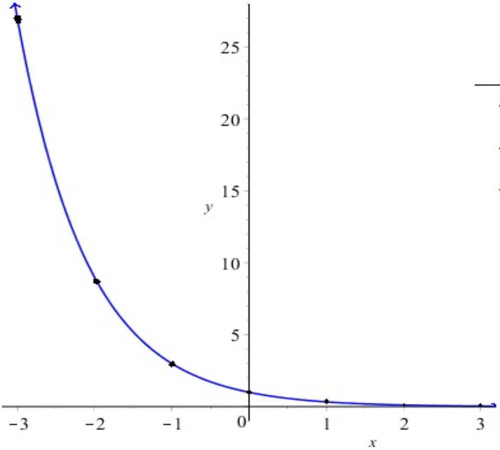
Graphs of Exponential Functions



x	$f(x) = 2^x$
-3	$1/8$
-2	$1/4$
-1	$1/2$
0	1
1	2
2	4
3	8

Figure: $f(x) = 2^x$ Note that the function is everywhere increasing. The x-axis is a horizontal asymptote.

Graphs of Exponential Functions



x	$f(x) = (1/3)^x$
-3	27
-2	9
-1	3
0	1
1	$1/3$
2	$1/9$
3	$1/27$

Figure: $f(x) = \left(\frac{1}{3}\right)^x$ Note that the function is everywhere decreasing. The x-axis is a horizontal asymptote.

Graphs of Exponential Functions

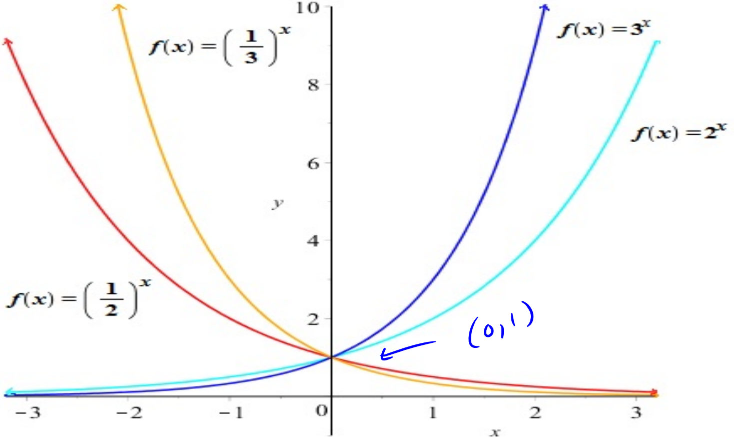
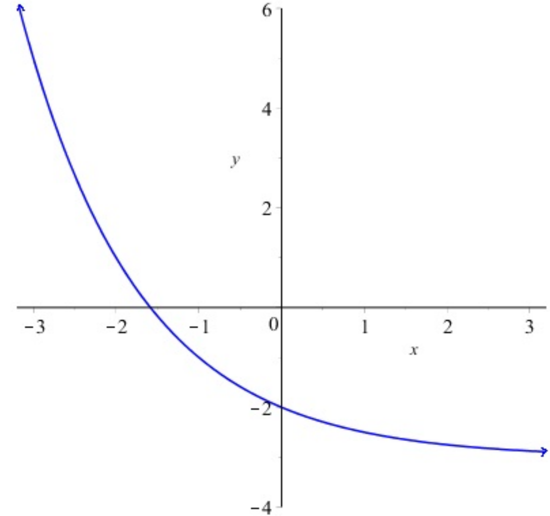


Figure: $f(x) = a^x$ is increasing if $a > 1$ and decreasing if $0 < a < 1$. The line $y = 0$ is a horizontal asymptote for every value of a. Each graph has y-intercept $(0, 1)$. Each graph is strictly above the x-axis.

Question

Given what you know about exponentials and graph transformations, the graph could be the plot of which function?



(a) $f(x) = \left(\frac{1}{2}\right)^x + 3$

(b) $f(x) = \left(\frac{1}{2}\right)^x - 3$

(c) $f(x) = 2^x + 3$

(d) $f(x) = 2^x - 3$

Question

The line $y = 0$ is a horizontal asymptote to the graph of $f(x) = \left(\frac{1}{2}\right)^x$.

True/False: The line $y = -3$ is a horizontal asymptote to the graph of $F(x) = \left(\frac{1}{2}\right)^x - 3$.

(a) True, and I'm confident.

(b) True, but I'm not confident.

(c) False, but I'm not confident.

(d) False, and I am confident.

as $x \to \infty$

$\left(\frac{1}{2}\right)^x \to 0$

so as $x \to \infty$

$\left(\frac{1}{2}\right)^x - 3 \to 0 - 3 = -3$

a^x and a^{-x}

$f(-x)$ is the reflection of $f(x)$ in the y-axis

Let's observe that by properties of exponents, we have

$$f(x) = a^{-x} = \frac{1}{a^x}.$$

So as we saw suggested in the graphs, the plots of

$$f_2(-x) = f_1(x) = 2^x \quad \text{and} \quad f_2(x) = \left(\frac{1}{2}\right)^x = f_1(-x)$$

$$= 2^{-x}$$

are reflections of one another in the y-axis.

The Favored Base

- From the graphs, we see that any base exponential can be obtained from any other base by stretching/shrinking and perhaps reflection in the y-axis.
- We can ask if there is a *natural* or prefered base.

The common base for *the* exponential function is the number

$$e \approx 2.718282.$$

The name e was given to this number by Leonhard Euler. It can be derived in several ways. One of these was discovered by Jacob Bernoulli in 1683 (this is credited as the first explicit derivation of the number).

Compounded Interest

Suppose P dollars is placed in an account that yields interest at an annual rate of r% compounded n times a year. The amount $A(t)$ in the account after t years is (with r in decimal form)

$$A(t) = P\left(1 + \frac{r}{n}\right)^{nt}.$$

Suppose we take a simple case of \$1 and an interest rate of 100%. What can be said about the yield in one year given different compounding schemes?

The Number e Derived

It was noted that the value $A = \left(1 + \frac{1}{n}\right)^n$ is increasing in n. Bernoulli's asked whether this grows without bound, or does it have a discernable *limit* value.

Number of Compounding periods n	Amount $= \left(1 + \frac{1}{n}\right)^n$
1 (compounded annually)	\$ 2
2 (compounded semiannually)	2.25
4 (compounded quarterly)	2.4414
12 (compounded monthly)	2.6130
365 (compounded daily)	2.7146
8760 (compounded hourly)	2.7181

Allowing n to grow without bound, the value approaches e. This number is irrational (Euler proved this well after Jacob Bernoulli's death.)

e^x and e^{-x}

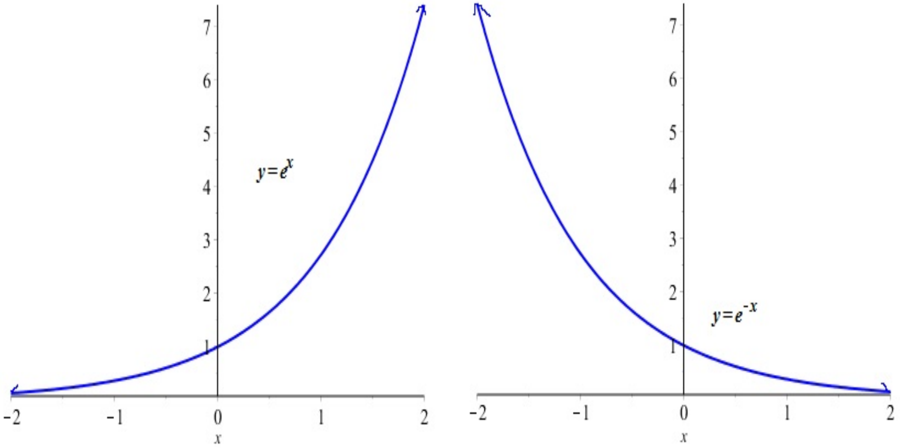
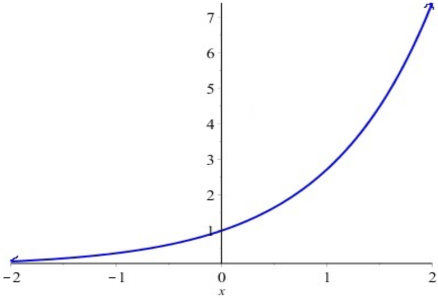


Figure: The exponential function $f(x) = e^x$ and the reciprocal function $g(x) = e^{-x}$ are among the most commonly used in applied mathematics. You should be able to produce these plots in your sleep!

Logarithms

Let's start with a **Question**.

True/False The exponential $f(x) = a^x$ is a one to one function.



Think horizontal line test

Figure: A plot of $f(x) = a^x$ for some $a > 1$.

Select (a) for True or (b) for False

Section 5.3: Inverse of an Exponential Function

Since $f(x) = a^x$ is one to one with domain $(-\infty, \infty)$ and range $(0, \infty)$, there must be an inverse function f^{-1} with

- domain $(0, \infty)$,
- range $(-\infty, \infty)$, and such that
- $f^{-1}(x) = y$ if and only if $a^y = x$ ← $f(y) = x$ since $f(y) = a^y$

For a given a, this inverse function is called the **logarithm function of base a.**

The Logarithm Function of Base a

Definition: Let $a > 0$ and $a \neq 1$. For $x > 0$ define $\log_a(x)$ as a number such that

$$\text{if} \quad y = \log_a(x) \quad \text{then} \quad x = a^y.$$

The function

$$F(x) = \log_a(x)$$

is called the **logarithm function of base** a. It has domain $(0, \infty)$, range $(-\infty, \infty)$, and if $f(x) = a^x$ then

$$F(x) = f^{-1}(x).$$

In particular

- $\log_a(a^x) = x$ for every real x, and
- $a^{\log_a(x)} = x$ for every $x > 0$.

Graph of Logarithms

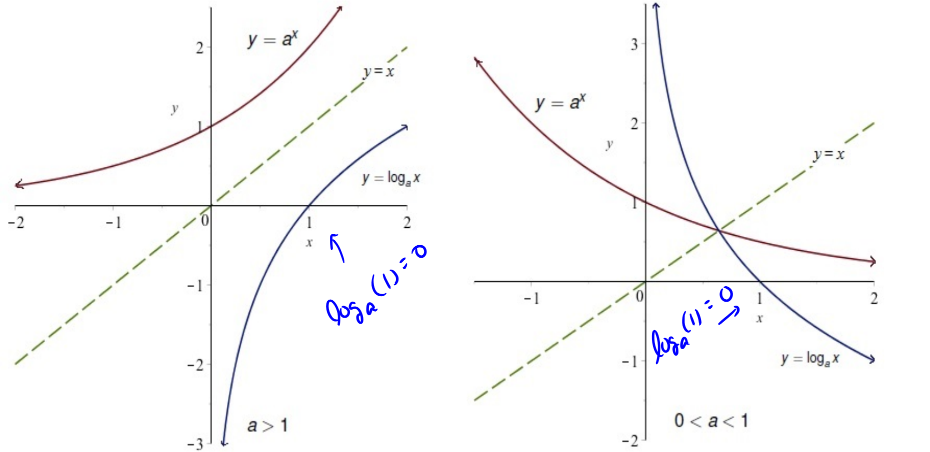


Figure: The graph of a logarithm can be obtained from the graph of an exponential by reflection in the line $y = x$. There are two cases depending on whether $0 < a < 1$ or $a > 1$.

Graph of Logarithms

The graph has some key properties. If $a > 1$, the graph of $F(x) = \log_a(x)$

- is increasing,
- has x-intercept $(1, 0)$,
- does not cross the y-axis,
- has vertical asymptote $x = 0$ (the y-axis) and tends to $-\infty$ as $x \to 0^+$
- grows without bound, albeit very slowly, as $x \to \infty$ (hence it has no horizontal asymptotes)

Graph of Logarithms

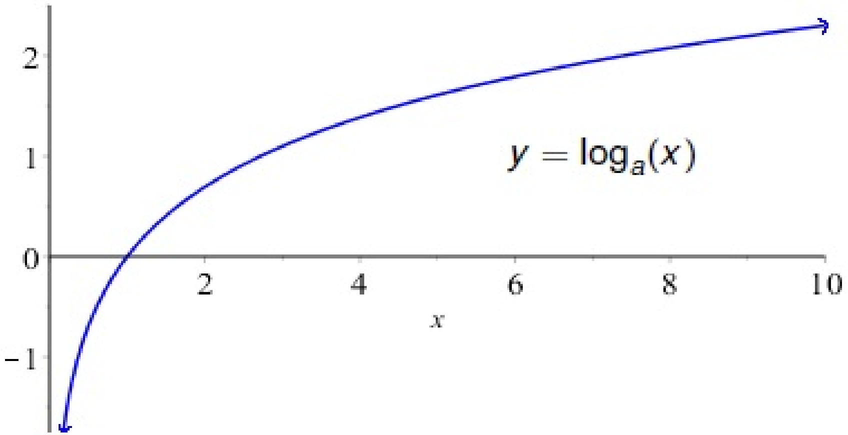


Figure: The graph of a logarithm with a base a where $a > 1$.

Graph of Logarithms

The graph has some key properties. If $0 < a < 1$, the graph of $F(x) = \log_a(x)$

- is decreasing,
- has x-intercept $(1, 0)$,
- does not cross the y-axis,
- has vertical asymptote $x = 0$ (the y-axis) and tends to ∞ as $x \to 0^+$
- goes down without bound, albeit very slowly, as $x \to \infty$ (hence it has no horizontal asymptotes)

Graph of Logarithms

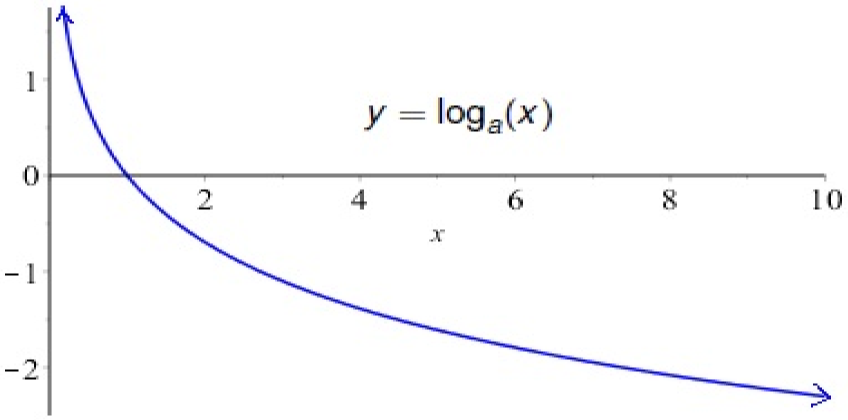


Figure: The graph of a logarithm with a base a where $0 < a < 1$

Evaluating Simple Logarithms

Use the fact that $y = \log_a(x)$ means $x = a^y$ to evaluate

(a) $\log_2(16) = 4$ $\qquad 16 = 2^y$

(b) $\log_{10}(0.001) = -3$ $\qquad 0.001 = 10^y$ since $0.001 = 10^{-3}$

(c) $\log_{1/2}(4) = -2$ $\qquad 4 = \left(\frac{1}{2}\right)^y = 2^{-y}$

(d) $\log_a(a^7) = 7$ $\qquad a^7 = a^y$

(e) $\log_\pi(1) = 0$ $\qquad 1 = \pi^y$

Question

Recall: $y = \log_a(x)$ means $x = a^y$.

If $e^t = 70$, then (which statement is true)

(a) $\log_e(t) = 70$

(b) $\log_{70}(e) = t$

(c) $\log_e(70) = t$

(d) $\log_t(e) = 70$

y

$e^t = 70$

a x

A Few Properties

- For every $a > 0$, $a^0 = 1$, hence $\log_a(1) = 0$.
- For every $a > 0$, $a^1 = a$, hence $\log_a(a) = 1$.
- For every $a > 0$, the expression $\log_a(0)$ is UNDEFINED!

Graphically: Every graph $y = \log_a(x)$ passes through the points $(1, 0)$ and $(a, 1)$. And

$$\begin{array}{cccc} \text{if } a > 1 & \log_a(x) \to -\infty & \text{as} & x \to 0^+ \\ \text{if } 0 < a < 1 & \log_a(x) \to \infty & \text{as} & x \to 0^+ \end{array}$$

Evaluating Logs with a Calculator

Most logarithms can't be evaluated by hand. For example, $\log_e(70) \approx 4.2485$. No one memorizes that! Calculators tend to have two built in logarithm functions.

- **Common Log** base 10 denoted as log (note there is no subscript), and
- **Natural Log** base *e* denoted[1] ln

In Calculus, you'll find that the prefered base is *e*—the natural log. Base 10 logs still have some use such as in the Richter scale used to measure earthquakes.

[1] The order LN instead of NL is probably due to the French name *le Logarithme Naturel* for this log.