Oct 8 Math 2306 sec. 53 Fall 2018

Section 9: Method of Undetermined Coefficients

The context here is linear, constant coefficient, nonhomogeneous equations

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_0 y = g(x)$$

where g comes from the restricted classes of functions

- polynomials,
- exponentials,
- sines and/or cosines,
- and products and sums of the above kinds of functions

Recall $y = y_c + y_p$, so we'll have to find both the complementary and the particular solutions!

We'll consider cases

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_0 y = g_1(x) + g_2(x) + \cdots + g_i(x)$$

Using superposition as needed, begin with assumption:

$$y_p = y_{p_1} + \cdots + y_{p_k}$$

where y_{p_i} has the same **general form** as $g_i(x)$.

Case I: y_p as first written has no part that duplicates the complementary solution y_c. Then this first form will suffice.

Case II: y_p has a term y_{p_i} that duplicates a term in the complementary solution y_c. Multiply **that term** (only the y_{p_i} at issue) by x^n, where n is the smallest positive integer that eliminates the duplication.

Case II Examples

Solve the ODE

constant coefficient

$$y'' - 2y' + y = -4e^x$$

← exponential

Find y_c: Char. equation $m^2 - 2m + 1 = 0$

$(m-1)^2 = 0 \Rightarrow m = 1$ repeated

$y_1 = e^x$, $y_2 = xe^x$

$y_c = c_1 e^x + c_2 x e^x$

Find y_p: $g(x) = -4e^x$

$y_p = Ae^x$ (won't work)

$y_p = Axe^x$ (won't work)

$y_p = Ax^2e^x$ Correct!

$$y'' - 2y' + y = -4e^x$$

$$y_p = Ax^2e^x$$

$$y_p' = Ax^2e^x + 2Axe^x$$

$$y_p'' = Ax^2e^x + 2Axe^x + 2Axe^x + 2Ae^x$$

$$y_p'' - 2y_p' + y_p =$$

$$Ax^2e^x + 4Axe^x + 2Ae^x - 2\left(Ax^2e^x + 2Axe^x\right) + Ax^2e^x = -4e^x$$

Collect like terms

$$x^2e^x\left(A - 2A + A\right) + xe^x\left(4A - 4A\right) + e^x\left(2A\right) = -4e^x$$

Matching $\Rightarrow$ $2A = -4$

$A = -2$

So $y_p = -2x^2 e^x$

$$\boxed{\begin{array}{c}\text{The general solution to the ODE is} \\ y = c_1 e^x + c_2 x e^x - 2x^2 e^x\end{array}}$$

Find the form of the particular soluition

$$y'' - 4y' + 4y = \sin(4x) + xe^{2x}$$

y_c : $m^2 - 4m + 4 = 0 \Rightarrow (m-2)^2 = 0 \Rightarrow$ $m = 2$ repeated

$y_1 = e^{2x}$, $y_2 = xe^{2x}$ $\quad y_c = C_1 e^{2x} + C_2 x e^{2x}$

y_p : let $g_1(x) = \sin(4x)$ and $g_2(x) = xe^{2x}$

$y_{p_1} = A\sin(4x) + B\cos(4x)$ (Doesn't match y_c this is correct)

$y_c = c_1 e^{2x} + c_2 x e^{2x}$ $\qquad$ $g_2(x) = x e^{2x}$

$y_{p_2} \cancel{=} (Cx+D)e^{2x} = Cxe^{2x} + De^{2x}$ $\qquad$ (not correct) matches y_c

$y_{p_2} \cancel{=} (Cx+D)xe^{2x} = Cx^2e^{2x} + Dxe^{2x}$ $\qquad$ (Dxe^{2x} still) matches y_c

$y_{p_2} = (Cx+D)x^2 e^{2x} = Cx^3 e^{2x} + Dx^2 e^{2x}$ $\qquad$ This is correct

$$y_p = A\sin(4x) + B\cos(4x) + (Cx^3 + Dx^2)e^{2x}$$

Find the form of the particular soluition

$$y''' - y'' + y' - y = \cos x + x^4$$

Find y_c : Charact. eqn

$$m^3 - m^2 + m - 1 = 0$$

$$m^2(m-1) + (m-1) = 0$$

$$(m^2+1)(m-1) = 0$$

$\alpha = 0$ $\beta = 1$

$$m - 1 = 0 \Rightarrow m = 1$$

or

$$m^2 + 1 = 0 \Rightarrow m^2 = -1 \Rightarrow m = \pm i = 0 \pm i$$

$$y_1 = e^{x}, \quad y_2 = e^{0x}\cos(1x), \quad y_3 = e^{0x}\sin(1x)$$

$$y_c = C_1 e^{x} + C_2 \cos(x) + C_3 \sin(x)$$

Let $g_1(x) = \cos x$ and $g_2(x) = x^4$

$$y_{p_1} \cancel{=} A\cos x + B\sin x$$ (not correct

$$y_{p_1} = (A\cos x + B\sin x)x = Ax\cos x + Bx\sin x$$

(this is correct)

$$y_{p_2} = Cx^4 + Dx^3 + Ex^2 + Fx + G$$ This is correct

y_p has the form

$$y_p = Ax\cos x + Bx\sin x + Cx^4 + Dx^3 + Ex^2 + Fx + G$$

Find the form of the particular soluition

$$y'' - 2y' + 5y = e^x + 7\sin(2x)$$

Find y_c: Charac. eqn

$m^2 - 2m + 5 = 0$

$m^2 - 2m + 1 - 1 + 5 = 0$

$(m-1)^2 + 4 = 0$

$(m-1)^2 = -4$

$m - 1 = \pm 2i$

$m = 1 \pm 2i$ $\alpha = 1$, $\beta = 2$

$y_1 = e^x \cos(2x) \qquad y_2 = e^x \sin(2x)$

$y_c = C_1 e^x \cos(2x) + C_2 e^x \sin(2x)$

Let $g_1(x) = e^x$, $g_2(x) = 7 \sin(2x)$

$y_{p_1} = A e^x$ This works

$y_{p_2} = B \sin(2x) + C \cos(2x)$ This works

$y_p = A e^x + B \sin(2x) + C \cos(2x)$