October 9 Math 2306 sec 51 Fall 2015

Section 4.6: Variation of Parameters

Solve the ODE

$$x^2y'' + xy' - 4y = \ln x,$$

given that $y_c = c_1x^2 + c_2x^{-2}$ is the complementary solution.

$y_1 = x^2, \quad y_2 = x^{-2}$

Standard form

$$y'' + \frac{1}{x}y' - \frac{4}{x^2}y = \frac{\ln x}{x^2}$$

$$W = \begin{vmatrix} x^2 & x^{-2} \\ 2x & -2x^{-3} \end{vmatrix}$$

$$g(x) = \frac{\ln x}{x^2}$$

$$= x^2(-2x^{-3}) - 2x(x^{-2}) = -2x^{-1} - 2x^{-1} = \frac{-4}{x}$$

$$u_1 = \int \frac{-y_2 g}{W}\,dx = \int \frac{-x^{-2}\,\frac{\ln x}{x^2}}{-4/x}\,dx = \frac{1}{4}\int x \cdot x^{-2} \cdot \ln x \cdot x^{-2}\,dx$$

$$= \frac{1}{4}\int x^{-3}\ln x\,dx$$

Int. by parts

$$\int w\,dv = wv - \int v\,dw$$

$$w = \ln x \qquad dw = \frac{1}{x}\,dx$$

$$v = \frac{-x^{-2}}{2} \qquad dv = x^{-3}\,dx$$

$$= \frac{1}{4}\left[\frac{-x^{-2}}{2}\ln x + \frac{1}{2}\int x^{-3}\,dx\right]$$

$$= \frac{1}{4}\left[\frac{-x^{-2}}{2}\ln x - \frac{1}{4}x^{-2}\right]$$

$$u_2 = \int \frac{y_1 g}{W}\,dx = \int \frac{x^2\left(\frac{\ln x}{x^2}\right)}{\frac{-4}{x}}\,dx = \frac{-1}{4}\int x \ln x\,dx$$

$$= -\frac{1}{4}\left[\frac{x^2}{2}\ln x - \frac{1}{2}\int x\,dx\right]$$

Int. by parts

$w = \ln x \qquad dw = \frac{1}{x}dx$

$v = \frac{x^2}{2} \qquad dv = x\,dx$

$$= \frac{-1}{4}\left[\frac{x^2}{2}\ln x - \frac{1}{4}x^2\right]$$

$$y_p = u_1 y_1 + u_2 y_2$$

$$= \frac{1}{4}\left[\frac{-x^{-2}}{2}\ln x - \frac{1}{4}x^{-2}\right]x^2 - \frac{1}{4}\left[\frac{x^2}{2}\ln x - \frac{1}{4}x^2\right]x^{-2}$$

$$= -\frac{1}{8}\ln x - \frac{1}{16} - \frac{1}{8}\ln x + \frac{1}{16}$$

$$= \frac{-2}{8}\ln x = \frac{-1}{4}\ln x$$

The general solution to the ODE is

$$y = C_1 x^2 + C_2 x^{-2} - \frac{1}{4}\ln x.$$

Solve the IVP

$$x^2y'' + xy' - 4y = \ln x, \quad y(1) = -1, \quad y'(1) = 0$$

From the last example

$$y = c_1 x^2 + c_2 x^{-2} - \frac{1}{4} \ln x$$

$$y' = 2c_1 x - 2c_2 x^{-3} - \frac{1}{4} \cdot \frac{1}{x}$$

$$y(1) = c_1 (1)^2 + c_2 (1)^{-2} - \frac{1}{4} \ln 1 = -1 \Rightarrow c_1 + c_2 = -1$$

$$y'(1) = 2c_1 (1) - 2c_2 (1)^{-3} - \frac{1}{4} \cdot \frac{1}{1} = 0 \Rightarrow 2c_1 - 2c_2 = \frac{1}{4}$$

$$2C_1 + 2C_2 = -2$$

$$2C_1 - 2C_2 = \frac{1}{4}$$

$$4C_1 = -\frac{7}{4} \Rightarrow C_1 = \frac{-7}{16} \qquad C_2 = -1 - C_1 = -1 + \frac{7}{16} = \frac{-9}{16}$$

The solution to the IVP is

$$\boxed{y = \frac{-7}{16}x^2 - \frac{9}{16}x^{-2} - \frac{1}{4}\ln x}$$

Section 4.9: Solving a System by Elimination

Consider the pair of differential equations

$$\begin{aligned} \frac{dx}{dt} + y &= t \\ x - 2\frac{dy}{dt} &= 1 \end{aligned}$$

This is a **linear system of differential equations.** It is linear in each of the two dependent variables (x and y). A **solution** would consist of a pair of functions $(x(t), y(t))$ that satisfied both equations simultaneously.

Linear System: IVP

A first order, constant coefficient system IVP has the form

$$\begin{aligned}\frac{dx}{dt} &= a_{11}x + a_{12}y + f(t), \quad x(t_0) = x_0\\ \frac{dy}{dt} &= a_{21}x + a_{22}y + g(t), \quad y(t_0) = y_0\end{aligned}$$

If $f(t) = g(t) = 0$, the system is *homogeneous*. Otherwise it is nonhomogeneous.

NOTE: there are two initial conditions, one for each dependent variable. Hence, the solution to the ODE part of the problem must be a 2-parameter family.

In general, for m dependent variables each satisfying an n^{th} order DE, there will be mn parameters. Here, $m = 2$ and $n = 1$.

Operator Notation

We'll introduce the following **operator** notation that will allow us to manipulate the system as though it were *algebraic*.
Define the operator D and the operator notation via

$$Dy = \frac{dy}{dt}, \quad D^2y = D(Dy) = \frac{d^2y}{dt^2}, \quad D^3y = D(D^2y) = \frac{d^3y}{dt^3}, \quad \cdots$$

Example:

Write the system of equations using the operator notation.

$$
\begin{aligned}
\frac{dx}{dt} + y &= t\\
x - 2\frac{dy}{dt} &= 1
\end{aligned}
$$

$$
\begin{aligned}
Dx + y &= t\\
x - 2Dy &= 1
\end{aligned}
$$

Example:

Write the system of equations using the operator notation.

$$\frac{d^2x}{dt^2} + 2x - \frac{dy}{dt} = \cos 2t$$
$$3\frac{dy}{dt} + \frac{dx}{dt} = -x + 4y - te^t$$

$$D^2x + 2x - Dy = \cos 2t$$

$$3Dy + Dx = -x + 4y - te^t$$

This is equivalent to

$$(D^2+2)x - Dy = \cos 2t$$

$$(D+1)x + (3D-4)y = -te^t$$