October 9 Math 2306 sec. 51 Fall 2019

Section 9: Method of Undetermined Coefficients

The context here is linear, constant coefficient, nonhomogeneous equations

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_0 y = g(x)$$

where g comes from the restricted classes of functions

- polynomials,
- exponentials,
- sines and/or cosines,
- and products and sums of the above kinds of functions

Recall $y = y_c + y_p$, so we'll have to find both the complementary and the particular solutions!

The Method

We're considering the ODE

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_0 y = g(x)$$

- Identify the type of function, g, on the right hand side.
- Assume that y_p has the same *form* as g with coefficients that will be determined by the ODE.
- Substitute your y_p into the ODE and match like terms.

Recall the Examples

$$y'' - 4y' + 4y = 8x + 1$$

We assumed that $y_p = Ax + B$ and ended up finding $y_p = 2x + \frac{9}{4}$.

$$y'' - 4y' + 4y = 6e^{-3x}$$

We assumed that $y_p = Ae^{-3x}$ and found that $y_p = \frac{6}{25}e^{-3x}$.

$$y'' - 4y' + 4y = 16x^2$$

We assumed that $y_p = Ax^2 + Bx + C$ and found that $y_p = 4x^2 + 8x + 6$.

General Forms & Rules of Thumb

- Always include sines and cosines together.
- Polynomial factors should account for all possible powers (up to the degree).
- Account for every type of term that could arise from taking derivatives.

Some other Examples

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_0 y = g(x)$$

Right Hand Side	First Assumed Form of y_p
$g(x) = 4\sin(2x)$	$y_p = A\sin(2x) + B\cos(2x)$
$g(x) = 2x^2 e^{-4x}$	$y_p = (Ax^2 + Bx + C)e^{-4x}$
$g(x) = -x\cos\left(\frac{\pi x}{2}\right)$	$y_p = (Ax + B)\cos\left(\frac{\pi x}{2}\right) + (Cx + D)\sin\left(\frac{\pi x}{2}\right)$
$g(x) = e^{2x}\cos(3x)$	$y_p = Ae^{2x}\cos(3x) + Be^{2x}\sin(3x)$

The Superposition Principle

Example: Determine the correct form of the particular solution using the method of undetermined coefficients for the ODE

$$y'' - 4y' + 4y = 6e^{-3x} + 16x^2$$

By superposition, we can look for y_p in the form $y_p = y_{p_1} + y_{p_2}$

Where y_{p_1} solves $y'' - 4y' + 4y = 6e^{-3x}$

$$y_{p_1} = Ae^{-3x}$$

And y_{p_2} solves $y'' - 4y' + 4y = 16x^2$

$$y_{p_2} = Bx^2 + Cx + D$$

Then the form of y_p is

$$y_p = Ae^{-3x} + Bx^2 + Cx + D$$

A Glitch!

$$y'' - y' = 3e^x$$

$g(x) = 3e^x$, so guess $y_p = Ae^x$.

Substitute: $y_p' = Ae^x$ and $y_p'' = Ae^x$

we need $y_p'' - y_p' = 3e^x$

$$Ae^x - Ae^x = 3e^x$$

$$0e^x = 3e^x$$

This requires $0 = 3$ which is always false!

We need to look at the associated homogeneous equation $y'' - y' = 0$. The characteristic equation is

$$m^2 - m = 0$$

$$m(m-1) = 0 \Rightarrow m = 0 \text{ or } m = 1$$

$$y_1 = e^{0x} = 1 \quad \text{and} \quad y_2 = e^{1x} = e^x$$

The complementary solution $y_c = c_1 + c_2 e^x$

Our guess Ae^x is part of y_c. Substitution into the left must give zero; we can't match to $3e^x$.

We'll consider cases

Using superposition as needed, begin with assumption:

$$y_p = y_{p_1} + \cdots + y_{p_k}$$

where y_{p_i} has the same **general form** as $g_i(x)$.

Case I: y_p as first written has no part that duplicates the complementary solution y_c. Then this first form will suffice.

Case II: y_p has a term y_{p_i} that duplicates a term in the complementary solution y_c. Multiply that term by x^n, where n is the smallest positive integer that eliminates the duplication.

Case II Examples

Find the general solution to the ODE.

$$y'' - y' = 3e^x$$

First, find y_c: $y_c'' - y_c' = 0$

we just found that $y_1 = 1$, $y_2 = e^x$

and $y_c = c_1 + c_2 e^x$.

Next, find y_p: $g(x) = 3e^x$

Starter guess: $y_p = Ae^x$ ✗ Duplicates $c_2 e^x$

modify $y_p = (Ae^x)x = Axe^x$ ✓ correct form

Substitute: $y_p = Axe^x$

$$y_p' = Ae^x + Axe^x$$

$$y_p'' = Ae^x + Ae^x + Axe^x = 2Ae^x + Axe^x$$

We require $y_p'' - y_p' = 3e^x$

$$2Ae^x + Axe^x - (Ae^x + Axe^x) = 3e^x$$

$$2Ae^x + Axe^x - Ae^x - Axe^x = 3e^x$$

$$e^x(2A - A) + xe^x(A - A) = 3e^x$$

$$Ae^x = 3e^x$$

$$A = 3$$

So $\quad y_p = 3x e^x$

The general solution is

$$y = c_1 + c_2 e^x + 3x e^x$$

$$y = y_c + y_p$$

Find the form of the particular solution

$$y'' - 4y' + 4y = \sin(4x) + xe^{2x}$$

First find y_c : $y'' - 4y' + 4y = 0$

Characteristic equation $m^2 - 4m + 4 = 0$

$(m-2)^2 = 0 \Rightarrow m = 2$ repeated

$y_1 = e^{2x}$, $y_2 = xe^{2x}$

$y_c = c_1 e^{2x} + c_2 x e^{2x}$

To find y_p, look for $y_p = y_{p_1} + y_{p_2}$

y_{p_1}: $y'' - 4y' + 4y = \sin(4x)$

$g(x) = \sin(4x)$ guess $y_{p_1} = A\sin(4x) + B\cos(4x)$ ✓

y_{p_2}: $y'' - 4y' + 4y = xe^{2x}$

$g(x) = xe^{2x}$ guess $y_{p_2} = (Cx+D)e^{2x}$

$= Cxe^{2x} + De^{2x}$ * Duplicates y_c

$y_{p_2} = (Cx+D)e^{2x} \cdot x$

$= Cx^2e^{2x} + Dxe^{2x}$ nope!

$y_{p_2} = (Cx+D)e^{2x} \cdot x^2$

$= Cx^3e^{2x} + Dx^2e^{2x}$ ✓

The correct form for y_p is

$$y_p = y_{p_1} + y_{p_2}$$

$$= A\sin(4x) + B\cos(4x) + Cx^3e^{2x} + Dx^2e^{2x}$$