October 9 Math 2306 sec 54 Fall 2015

Section 4.6: Variation of Parameters

Solve the ODE

$$x^2y'' + xy' - 4y = \ln x,$$

given that $y_c = c_1x^2 + c_2x^{-2}$ is the complementary solution.

$y_1 = x^2$, $y_2 = x^{-2}$

Standard form:

$$y'' + \frac{1}{x}y' - \frac{4}{x^2}y = \frac{\ln x}{x^2}$$

$$g(x) = \frac{\ln x}{x^2}$$

$$W = \begin{vmatrix} x^2 & x^{-2} \\ 2x & -2x^{-3} \end{vmatrix}$$

$$= x^2(-2x^{-3}) - 2x(x^{-2}) = -4x^{-1} = \frac{-4}{x}$$

$$u_1 = \int \frac{-y_2 g}{W}\,dx = \int \frac{-x^{-2}\left(\frac{\ln x}{x^2}\right)}{-4/x}\,dx = \frac{1}{4}\int x\cdot x^{-2}\ln x\cdot x^{-2}\,dx$$

$$= \frac{1}{4}\int x^{-3}\ln x\,dx$$

Int. by parts

$$\int w\,dv = wv - \int v\,dw$$

$$= \frac{1}{4}\left[\frac{-x^{-2}}{2}\ln x + \frac{1}{2}\int x^{-3}\,dx\right]$$

$$w = \ln x \qquad dw = \frac{1}{x}\,dx$$

$$v = \frac{-x^{-2}}{2} \qquad dv = x^{-3}\,dx$$

$$= \frac{1}{4}\left[\frac{-x^{-2}}{2}\ln x - \frac{1}{4}x^{-2}\right]$$

$$u_2 = \int \frac{y_1 g}{W}\,dx = \int \frac{x^2\left(\frac{\ln x}{x^2}\right)}{-4/x}\,dx =$$

$$= \frac{-1}{4}\int x \ln x \; dx$$

Int. by parts

$w = \ln x \qquad dw = \frac{1}{x}\,dx$

$$= \frac{-1}{4}\left[\frac{x^2}{2}\ln x - \frac{1}{2}\int x\,dx\right]$$

$v = \frac{x^2}{2} \qquad dv = x\,dx$

$$= \frac{-1}{4}\left[\frac{x^2}{2}\ln x - \frac{1}{4}x^2\right]$$

$$y_p = u_1 y_1 + u_2 y_2$$

$$= \frac{1}{4}\left[\frac{-x^{-2}}{2}\ln x - \frac{x^{-2}}{4}\right]x^2 - \frac{1}{4}\left[\frac{x^2}{2}\ln x - \frac{x^2}{4}\right]x^{-2}$$

$$= -\frac{1}{8}\ln x - \frac{1}{16} - \frac{1}{8}\ln x + \frac{1}{16}$$

$$= \frac{-2}{8}\ln x = \frac{-1}{4}\ln x$$

The general solution to the ODE is

$$y = c_1 x^2 + c_2 x^{-2} - \frac{1}{4}\ln x.$$

Solve the IVP

$$x^2y'' + xy' - 4y = \ln x, \quad y(1) = -1, \quad y'(1) = 0$$

From before

$$y = C_1 x^2 + C_2 x^{-2} - \frac{1}{4}\ln x$$

$$y' = 2C_1 x - 2C_2 x^{-3} - \frac{1}{4}\cdot\frac{1}{x}$$

$$y(1) = C_1 (1)^2 + C_2 (1)^{-2} - \frac{1}{4}\ln 1 = -1 \implies C_1 + C_2 = -1$$

$$y'(1) = 2C_1(1) - 2C_2(1)^{-3} - \frac{1}{4}\cdot\frac{1}{1} = 0 \implies 2C_1 - 2C_2 = \frac{1}{4}$$

$$2c_1 + 2c_2 = -2$$
$$2c_1 - 2c_2 = \frac{1}{4}$$
$$4c_1 = \frac{-7}{4} \Rightarrow c_1 = \frac{-7}{16}$$

$$c_2 = -1 - c_1$$
$$= -1 + \frac{7}{16} = \frac{-9}{16}$$

The solution to the IVP is

$$y = \frac{-7}{16} x^2 - \frac{9}{16} x^{-2} - \frac{1}{4} \ln x$$

Section 4.9: Solving a System by Elimination

Consider the pair of differential equations

$$\begin{aligned} \frac{dx}{dt} + y &= t \\ x - 2\frac{dy}{dt} &= 1 \end{aligned}$$

This is a **linear system of differential equations.** It is linear in each of the two dependent variables (x and y). A **solution** would consist of a pair of functions $(x(t), y(t))$ that satisfied both equations simultaneously.

Linear System: IVP

A first order, constant coefficient system IVP has the form

$$\begin{aligned}\frac{dx}{dt} &= a_{11}x + a_{12}y + f(t), \quad x(t_0) = x_0\\ \frac{dy}{dt} &= a_{21}x + a_{22}y + g(t), \quad y(t_0) = y_0\end{aligned}$$

If $f(t) = g(t) = 0$, the system is *homogeneous*. Otherwise it is nonhomogeneous.

NOTE: there are two initial conditions, one for each dependent variable. Hence, the solution to the ODE part of the problem must be a 2-parameter family.

In general, for m dependent variables each satisfying an n^{th} order DE, there will be mn parameters. Here, $m = 2$ and $n = 1$.

Operator Notation

We'll introduce the following **operator** notation that will allow us to manipulate the system as though it were *algebraic*.
Define the operator D and the operator notation via

$$Dy = \frac{dy}{dt}, \quad D^2y = D(Dy) = \frac{d^2y}{dt^2}, \quad D^3y = D(D^2y) = \frac{d^3y}{dt^3}, \quad \cdots$$

Example:

Write the system of equations using the operator notation.

$$\begin{aligned} \frac{dx}{dt} + y &= t \\ x - 2\frac{dy}{dt} &= 1 \end{aligned}$$

$$\begin{aligned} Dx + y &= t \\ x - 2Dy &= 1 \end{aligned}$$

Example:

Write the system of equations using the operator notation.

$$\begin{aligned}\frac{d^2x}{dt^2} + 2x - \frac{dy}{dt} &= \cos 2t\\ 3\frac{dy}{dt} + \frac{dx}{dt} &= -x + 4y - te^t\end{aligned}$$

$$\begin{aligned}D^2x + 2x - Dy &= \cos 2t\\ 3Dy + Dx &= -x + 4y - te^t\end{aligned}$$

This is equivalent to

$$\begin{aligned}(D^2+2)x - Dy &= \cos 2t\\ (D+1)x + (3D-4)y &= -te^t\end{aligned}$$

Solving a System by Elimination

Remark: The current method is for linear systems with constant coefficients only.

- Write the system using the operator notation. Line up like variables so that the system appears as an algebraic system.
- Eliminate variables using standard operations. Keep in mind that "multiplication" by D is differentiation.
- Obtain an equation (or equations) in each variable separately, and solve using any applicable method.
- Use back substitution as needed to obtain solutions for all dependent variables.

Solve the System by Elimination

$$\frac{dx}{dt} = 4x + 7y \qquad Dx = 4x + 7y \Rightarrow Dx - 4x - 7y = 0$$

$$\frac{dy}{dt} = x - 2y \qquad Dy = x - 2y \Rightarrow -x + Dy + 2y = 0$$

$$(D-4)x - 7y = 0$$

$$-x + (D+2)y = 0$$

* "Multiply" equation 2 by $D-4$ and add

$$(D-4)x - 7y = 0$$

$$-(D-4)x + (D-4)(D+2)y = (D-4)\cdot 0 = 0$$

$$-7y+(D-4)(D+2)\,y=0$$

$$\left[-7+D^2-4D+2D-8\right]y=0$$

$$(D^2-2D-15)\,y=0 \quad\Rightarrow\quad y''-2y'-15y=0$$

2^{nd} order linear constant coeff. eqn (see 4.3)

$$m^2-2m-15=0 \quad\Rightarrow\quad (m-5)(m+3)=0$$

$$m_1=5,\quad m_2=-3$$

$$\boxed{y=C_1e^{5t}+C_2e^{-3t}}$$

We're not done. We still need to find $x(t)$.

We'll pick this up on Monday.