

## Population Modeling

### Calculus II Project

There are a number of applications in biology that are primarily concerned with tracking populations—of bacteria, foxes and rabbits, of game fish, of humans, and on it goes. One might be interested in tracking a single population that is affected only by environmental phenomena that are considered constant (see for example Malthusian or Logistic growth models). Other applications may involve tracking populations of two or more interrelated species (predator/prey or competing species).

In this project, you will consider some simple population models in the form of differential equations. The analysis of such models can include actually finding explicit (or implicit) solutions, but interesting conclusions can also be drawn from the equations themselves. I strongly recommend use of appropriate technology. A particularly wonderful set of programs for this purpose have been created by John Polking of Rice University. Java versions written by Joel Castellanos (U. of New Mexico) are available on this dfield and pplane webpage for free immediate use (chosed the .jar downloads).

### Carry out the following activities.

**A.** A simple assumption in population dynamics is that the rate at which a population changes is directly proportional to the population (Malthusian model). This captures reasonably well the observation that organisms arise via reproduction. If the number of organisms<sup>1</sup>  $P$  at time  $t$  have a net (birth minus death) reproductive rate  $k$ , this gives the model

$$\frac{dP}{dt} = kP.$$

Consider a population of bacteria with initial count of  $P(0) = 5000$  and let  $t$  be in hours. Assuming the reproductive rate  $k = 0.024$ ,

- Determine the appropriate units for the constant  $k$ .
- Solve the differential equation subject to the initial count to find  $P(t)$  for all future time. (Read section 9.3 in Stewart on separable differential equations. It will be useful throughout this project.)
- Let  $\tau$  be the time required for the population to double. Find  $\tau$ .
- How many days does it take for the population to reach one million?

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<sup>1</sup>To ensure that  $P$  makes sense as a real-valued variable, it is probably better to consider  $P$  as a population density than as a number of individuals.

What strengths and weaknesses can you identify for this model?

**B.** Suppose that environmental factors limit the population. This may be captured by assuming that a population's rate of change is jointly proportional to the current population and the difference between the population and the maximum sustainable population. Mathematically,

$$\frac{dP}{dt} = rP(M - P), \quad \text{where } M \text{ is the maximum population.}$$

- Assuming  $P$  is number of individuals, determine the correct units for  $M$  and  $r$ .
- Solve the differential equation subject to the initial population  $P(0) = P_0$ .
- Determine the long term behavior of the population  $\lim_{t \rightarrow \infty} P(t)$  (if  $P_0 = 0$ ; if  $0 < P_0 < M$ ; if  $P_0 > M$ ).
- Explain the long term behavior found.
- Determine the population at which the rate of change of population is maximum.

Compare this model (the Logistic model) to the Malthusian.

**C.** An *equilibrium* solution to a differential equation is a constant solution. When the independent variable represents time, equilibrium solutions usually give an indication of the expected long time behavior of the dependent variable. Some equilibria (plural of equilibrium) are likely to be observed; others are not. This can be related to a property referred to as *stability*.

Find all of the equilibria for the logistic model from part **B**. Show that for initial population  $P_0 > 0$ , one of these constant solutions will be observed (at least in the limit sense as  $t \rightarrow \infty$ ). Explain why the population will increase (or decrease depending on  $P_0$ ) toward this equilibrium. Why doesn't the population go to zero?

**D.** Consider a modification of the logistic equation that accounts for harvesting (e.g. due to fishing).

$$\frac{dP}{dt} = rP(M - P) - h, \quad \text{where } h \text{ is a constant harvesting parameter.}$$

- Determine the appropriate units for the parameter  $h$ .
- Take the simple case  $r = 1$ . Solve the differential equation subject to the initial condition  $P(0) = P_0$  under the assumption  $h < \frac{M^2}{4}$ .
- Keeping  $r = 1$ , determine the two equilibria for the case  $h < \frac{M^2}{4}$ .

- Using field or another method (technological or purely mathematical), determine which of the equilibria is expected to correspond to long time population and which is not.
- Suppose  $h > \frac{M^2}{4}$ . Are there any equilibria in this case? Does this change the types of solutions?

**E.** Suppose that a particular organism's breeding behavior requires a minimum population to ensure sufficient reproduction. The logistic model may be modified to account for a minimum population  $m$  as

$$\frac{dP}{dt} = r(M - P)(P - m), \quad 0 < m < M.$$

- Show that the population is increasing if  $m < P < M$  and decreasing if  $P < m$  or  $P > M$ .
- Determine the equilibria for this equation. What is the expected long time behavior for each case on the initial condition

$$(a) P_0 = \frac{m}{2}, \quad (b) P_0 = m, \quad (c) P_0 = \frac{m + M}{2}, \quad (d) P_0 = 2M.$$

- Take the simple case  $r = 1$ . Solve the differential equation subject to  $P(0) = P_0$ .
- Show that if  $0 < P_0 < m$ , the species will become extinct. (There should be a term in your numerator that becomes zero at some finite  $t$ .)

**F.** Suppose we have a population of prey animals  $x$  and a population of predator animals  $y$ . Assuming that the prey will reproduce in Malthusian form in the absence of predation, predator-prey interactions decrease prey and increase predator populations, and predators will decay in Malthusian form without prey, one model for the two populations is the *Lotka-Volterra* model

$$\frac{dx}{dt} = ax - bxy, \tag{1}$$

$$\frac{dy}{dt} = dxy - cy. \tag{2}$$

Each of the parameters  $a, b, c$  and  $d$  are nonnegative. For  $x$  in number of prey animals and  $y$  in number of predator animals, determine the appropriate units for each of  $a, b, c$ , and  $d$ .

An equilibrium solution for such a system would be one for which both  $x = \text{const}$  and  $y = \text{const}$ , simultaneously. Find all possible equilibrium solutions to this system. The java program pplane includes this model in its gallery (from the PPLANE Equation Window, choose Gallery, then Predator/Prey). Explore the model using pplane or similar technology. Vary the parameter values and look at the resulting trajectories. From the Phase Plane window (in pplane), you can see graphs of  $x(t)$  and

$y(t)$  (choose Graph, then  $x$  vs.  $t$ ,  $y$  vs.  $t$ , or both). Explain the behavior from a population perspective.

One approach to analyzing such a system is to consider  $t$  an intermediate parameter and use the result

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

to obtain a single differential equation in  $x$  and  $y$ . For this system, the result is a separable equation. Do this, here and solve the resulting separable equation to obtain an implicit solution characterizing the populations.