Solutions to Practice for Exam I (Ritter) MATH 3260 Fall 2017

Sections Covered: 1.1, 1.2, 1.3, 1.4, 1.5, 1.7, 1.8, 1.9

This practice exam is intended to give you a rough idea of the types of problems you can expect to encounter. **Nothing else is intended or implied.**

(1) Row reduce the matrix to reduced echelon form (by hand). Determine the pivot positions and pivot columns in the matrix.

$$\begin{bmatrix} 2 & 4 & 6 & 8 \\ 4 & 6 & 8 & 10 \\ 6 & 8 & 10 & 4 \end{bmatrix}$$

The row reduction can be completed in about 5 stages leading to an rref

$$\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The pivot positions are the (1, 1), (2, 2) and (3, 4) positions, and columns 1, 2 and 4 are pivot columns.

(2) Find the solution set of the homogeneous linear system. Express the solution in parametric vector form.

$$\begin{array}{ccccccccc} 2x_1 & + & 4x_2 & + & 6x_3 & + & 8x_4 & = & 0 \\ 4x_1 & + & 6x_2 & + & 8x_3 & + & 10x_4 & = & 0 \\ 6x_1 & + & 8x_2 & + & 10x_3 & + & 4x_4 & = & 0 \end{array}$$

Reading the results off of the rref from problem (1), the solutions

$$\mathbf{x} = t \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \quad -\infty < t < \infty.$$

(3) Determine the value(s) of h and k such that the system has

(a) exactly one solution, $h \neq 9$, k anything

(b) infinitely many solutions, $h = 9$, $k = 6$

(c) no solutions. $h = 9$, $k \neq 6$

$$\begin{array}{rcrcr} x_1 & + & 3x_2 & = & 2 \\ 3x_1 & + & hx_2 & = & k \end{array}$$

(4) Find the solution set for each linear system of equations. Use an augmented matrix with row operations. You may express the solution parametrically ($x_1 =$blah, $x_2 =$ blah blah etc.) or in parametric vector form, your choice.

(a) $$\begin{array}{rcrcrcr} 2x_1 & - & 2x_2 & + & x_3 & = & 6 \\ x_1 & + & x_2 & - & x_3 & = & -2 \\ & & x_2 & + & 3x_3 & = & 5 \end{array} \qquad \mathbf{x} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$

(b) $$\begin{array}{rcrcrcr} 2x_1 & + & 5x_2 & - & 3x_3 & = & 6 \\ x_1 & - & x_2 & & & = & 2 \\ 3x_1 & + & 4x_2 & - & 3x_3 & = & 8 \end{array} \qquad \mathbf{x} = \begin{bmatrix} \frac{16}{7} \\ \frac{2}{7} \\ 0 \end{bmatrix} + \frac{x_3}{7} \begin{bmatrix} 3 \\ 3 \\ 7 \end{bmatrix}$$

(c) $$\begin{array}{rcrcrcr} x_1 & + & 2x_2 & - & 4x_3 & = & 0 \\ 2x_1 & + & 4x_2 & - & 8x_3 & = & 8 \end{array}$$

The system is inconsistent.

(5) Consider the given set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} -1 \\ 3 \\ 0 \end{bmatrix}, \quad \mathbf{v}_4 = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}.$$

(a) Determine if $\mathbf{b} = (1, 3, 1)$ is in Span$\{\mathbf{v}_1, \mathbf{v}_2\}$. If so, identify the weights. [1] Yes, $\mathbf{b} = \mathbf{v}_1 + \frac{3}{2}\mathbf{v}_2$

(b) Determine if $\mathbf{b} = (2, -2, -6)$ is in Span$\{\mathbf{v}_1, \mathbf{v}_2\}$. If so, identify the weights. Yes, $\mathbf{b} = 2\mathbf{v}_1 - \mathbf{v}_2$

(c) Determine if the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent. If not, find a linear dependence relation. They are linearly independent.

(d) Determine if the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ is linearly independent. If not, find a linear dependence relation. $6\mathbf{v}_1 + 6\mathbf{v}_2 - 4\mathbf{v}_3 - 5\mathbf{v}_4 = \mathbf{0}$ They are dependent.

(e) Does the matrix equation $A\mathbf{x} = \mathbf{0}$ have a nontrivial solution if $A = [\mathbf{v}_1\ \mathbf{v}_3\ \mathbf{v}_4]$? Justify your answer. No. If you reduce A to rref, you get I_3.

(6) Determine whether the transformation $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^4$ is a linear transformation. Justify your conclusion. No. For one thing, $T(\mathbf{0}) \neq \mathbf{0}$.

$$T(x_1, x_2, x_3) = (x_1 + 1, 0, 0, x_2 - x_3)$$

(7) Show that T is a linear transformation by finding a matrix that implements the mapping. (The terms x_1, x_2 etc. are entries in a vector as opposed to vectors themselves.) The standard matrix $A = \begin{bmatrix} 1 & -2 \\ 0 & 3 \\ 0 & 0 \\ 4 & 1 \end{bmatrix}$

$$T(x_1, x_2) = (x_1 - 2x_2, 3x_2, 0, 4x_1 + x_2)$$

[1] The vector $\mathbf{b}$ is equivalent to the column

$$\begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$$

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(8) Show that if T is a linear transformation, then it is necessarily true that $T(\mathbf{0}) = \mathbf{0}$. There are several acceptable arguments. One can be made by taking any $\mathbf{x}$ in the domain. By definition of scalar multiplication, $\mathbf{0} = 0\mathbf{x}$ (the zero vector in the domain of T). Then using the linearity properties, $T(\mathbf{0}) = T(0\mathbf{x}) = 0T(\mathbf{x}) = \mathbf{0}$ where the final term is the zero vector in the codomain.

(9) Let $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a linear tranformation defined by

$$T(x_1, x_2, x_3) = (x_1 - 3x_3, 2x_1 + 2x_2).$$

(a) Identify the domain and codomain. $\mathbb{R}^3$ and $\mathbb{R}^2$, respectively

(b) Determine if T is one to one. No. Note that $T(3, -3, 1) = (0, 0)$.

(c) Determine if T is onto. Yes. Show for example that $\{T(\mathbf{e}_2), T(\mathbf{e}_3)\}$ spans $\mathbb{R}^2$.

(d) Is $(1, 1)$ in the domain of T? If so, find its image. Is $(1, 1)$ in the range of T? Why or why not? No to the first since it's not in $\mathbb{R}^3$. Yes to the second by virtue of the answer to part (c) above.

(10) Let A, B, and C be the coefficient matrices[2] for the linear systems in problem (4) (of this review), respectively. Compute each expression if it exists. If it doesn't exist, state why.

(a) $A + 2B$ $\quad \begin{bmatrix} 6 & 8 & -5 \\ 3 & -1 & -1 \\ 6 & 9 & -3 \end{bmatrix}$

(b) A^T $\quad \begin{bmatrix} 2 & 1 & 0 \\ -2 & 1 & 1 \\ 1 & -1 & 3 \end{bmatrix}$

[2] A is the coeff. matrix for part (a), B for part (b), etc.

(c) $B+C$ This is undefined since the matrices are not of the same size.

(d) AC This is undefined. C does not have the same number of rows as A does columns.

(e) AC^T

$$\begin{bmatrix} -6 & -12 \\ 7 & 14 \\ -10 & -20 \end{bmatrix}$$

(f) C^TC

$$\begin{bmatrix} 5 & 10 & -20 \\ 10 & 20 & -40 \\ -20 & -40 & 80 \end{bmatrix}$$

As an aside, note that $(C^TC)^T = C^TC$ as expected. There is a necessary symmetry to the entries of this product.