Sections Covered: 2.2, 2.3, 3.1, 3.2, 4.1, 4.2, 4.3, 4.4, 4.5, 4.6

This practice exam is intended to give you a rough idea of the types of problems you can expect to encounter. Nothing else is intended or implied.

(1) Evaluate the determinant of each matrix.

(a) \[
\begin{vmatrix}
3 & 0 & 0 & 1 \\
0 & 2 & 0 & 0 \\
1 & -1 & 0 & 1 \\
0 & 4 & 0 & 2
\end{vmatrix}
\]

(b) \[
\begin{vmatrix}
2 & 1 & -1 \\
3 & 2 & 0 \\
-2 & 0 & -2
\end{vmatrix}
\]

(c) \[
\begin{vmatrix}
2 & 1 & -1 \\
0 & k & 3 \\
0 & 0 & k - 1
\end{vmatrix}
\]

(2) Find all values of \(x\) such that \(\det(A) = 0\) where

\[
A = \begin{bmatrix} x - 1 & 3 \\ 2 & x \end{bmatrix}
\]

(3) Solve the following system of equations using a matrix inverse.

\[
\begin{aligned}
2x_1 - 4x_2 &= 5 \\
-3x_1 + 2x_2 &= 0
\end{aligned}
\]

(4) Use the row reduction algorithm to find the inverse matrix if it exists.

\[
\begin{bmatrix}
1 & 1 & 1 \\
0 & 1 & 1 \\
0 & 0 & 1
\end{bmatrix}
\]

(5) Determine if the given set is a subspace of \(\mathbb{R}^3\).

(a) \(S = \{(x_1, x_2, x_3) | x_1 = x_2 + x_3\}\)

(b) \(S = \{(x_1, x_2, x_3) | x_1 - x_3 = 1\}\)
(6) Let \( P_2 \) denote the set of all polynomials of degree less than or equal to 2. Determine if the following sets are subspaces of \( P_2 \). (The symbol "\( \in \)" means "is an element of".)

(a) \( S = \{at^2+bt+c | a > 0\} \)

(b) \( S = \{p(t) \in P_2 | p(1) = 0 \text{ and } p(0) = 0\} \)

(7) Find a matrix \( A \) such that the given set is \( \text{Col}A \).

\[
\left\{ \begin{bmatrix} b - c \\ 2b + c + d \\ 5c - 4d \\ d \end{bmatrix} : b, c, d \in \mathbb{R} \right\}
\]

(8) Find a spanning set for \( \text{Nul}A \) and a spanning set for \( \text{Col}A \) where

\[
A = \begin{bmatrix} 3 & 6 & 1 & -6 \\ 2 & 4 & 1 & -5 \end{bmatrix}
\]

(9) The matrices \( A \) and \( B \) are row equivalent. Use this information to find bases for \( \text{Nul}A \), \( \text{Col}A \), and \( \text{Row}A \). Determine both \( \text{rank}A \) and \( \text{dim}(\text{Nul}A) \).

(a) \[
A = \begin{bmatrix} 1 & 3 & 4 & -1 & 2 \\ 2 & 6 & 6 & 0 & -3 \\ 3 & 9 & 3 & 6 & -3 \\ 3 & 9 & 0 & 9 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 3 & 4 & -1 & 2 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}
\]
(b) \( A = \begin{bmatrix}
1 & 1 & -2 & 0 & 1 & -2 \\
1 & 2 & -3 & 0 & -2 & -3 \\
1 & -1 & 0 & 0 & 1 & 6 \\
1 & -2 & 2 & 1 & -3 & 0 \\
1 & -2 & 1 & 0 & 2 & -1
\end{bmatrix} \), \( B = \begin{bmatrix}
1 & 1 & -2 & 0 & 1 & -2 \\
0 & 1 & -1 & 0 & -3 & -1 \\
0 & 0 & 1 & 1 & -13 & -1 \\
0 & 0 & 0 & 0 & 1 & -1 \\
0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix} \)

(10) Show that the given set is a basis for \( \mathbb{R}^2 \). Determine the change of coordinates matrix \( P_B \) and its inverse (use the order presented here). Then use this to find the indicated coordinate vectors.

\[ B = \left\{ \begin{bmatrix} 5 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \end{bmatrix} \right\} \]

Determine \([x]_B\) for

(a) \( x = \begin{bmatrix} 5 \\ 1 \end{bmatrix} \), (b) \( x = \begin{bmatrix} 2 \\ 2 \end{bmatrix} \), (c) \( x = \begin{bmatrix} -2 \\ 3 \end{bmatrix} \)

(11) Determine the dimension of the indicated vector space.

(a) The subspace of \( \mathbb{R}^4 \) of vectors whose components sum to zero.

(b) The subspace of \( \mathbb{P}_4 \) consisting of polynomials of the form \( p(t) = at^4 + b(t^2 - t) \), for real numbers \( a \) and \( b \).

(c) The null space of a \( 5 \times 8 \) matrix with a rank of 4.

(d) The row space of an \( m \times n \) matrix whose null space has dimension 6. (It may be necessary to express the answer in terms of \( m \) or \( n \) or both.)
Vector Space Axioms

A nonempty set $V$ together, with operations called *addition* and *scalar multiplication*, is a vector space if the following properties hold for every $u$, $v$, and $w$ in $V$ and scalars $c$ and $d$ in $\mathbb{R}$.

1. $u + v$ is in $V$
2. $u + v = v + u$
3. $(u + v) + w = u + (v + w)$
4. There is an element denoted $0$ in $V$ such that $u + 0 = u$ for every $u$ in $V$.
5. For each element $u$ in $V$ there is an element denoted $-u$ such that $u + (-u) = 0$
6. $cu$ is in $V$ for each scalar $c$
7. $(cd)u = c(du)$
8. $(c + d)u = cu + du$
9. $c(u + v) = cu + cv$
10. $1u = u$