Solutions to Review for Exam III

MATH 1113 sections 51 & 52 Fall 2018

Sections Covered: 2.2 (diff. quotient), 2.1 (piecewise fnct), 5.1, 5.2, 5.3, 5.4, 5.5, 6.1 & 6.2

Calculator Policy: Calculator use may be allowed on part of the exam. When instructions call for an **exact** solution, that indicates that a decimal approximation will not be accepted.

This review is provided as a courtesy to give some idea of what material is covered. Nothing else is intended or implied.

(1) Let $f(x) = \begin{cases} 2x-1, & -2 \leq x \leq 1 \\ 0, & 1 < x < 2 \\ 1, & x = 2 \\ 5 - x^2, & 2 < x \leq 3 \end{cases}$ Evaluate each of the following if possible. If a quantity doesn't exist, you can write "DNE." Where applicable, assume that $0 < h < 0.1$.

(a) $f(0) = -1$

(b) $f\left(\frac{5}{2}\right) = -\frac{5}{4}$

(c) $f(4)$ DNE (not in domain)

(d) $f(1+h) = 0$

(e) $f(1-h) = 1 - 2h$

(f) $f(2+h) = 1 - 4h - h^2$

(2) Provide a sketch of each piecewise defined function. Identify the domain and range of each function. Hand drawn plots are at the end of this set of solutions.

(a) $f(x) = \begin{cases} 2x-1, & -2 \leq x \leq 1 \\ 0, & 1 < x < 2 \\ 1, & x = 2 \\ 5 - x^2, & 2 < x \leq 3 \end{cases}$

(b) $g(x) = \begin{cases} x+2, & -3 < x < -1 \\ x^2, & -1 < x < 1 \\ 3 - x, & 1 \leq x \leq 3 \end{cases}$

(c) $h(x) = \begin{cases} e^{-x}, & -1 \leq x \leq 0 \\ \ln(x+1), & 0 < x \end{cases}$

(3) For each function and given value for a, evaluate the difference quotient $\dfrac{f(a+h) - f(a)}{h}$. Simplify your answer.

(a) $f(x) = 2x^2 - x, \quad$ for $\quad a = -1$

$$\frac{f(a+h)-f(a)}{h} = \frac{f(-1+h)-f(-1)}{h} = \frac{2(-1+h)^2-(-1+h)-(2(-1)^2-(-1))}{h} =$$

$$= \frac{2(1-2h+h^2)+1-h-3}{h} = \frac{-5h+2h^2}{h} = \frac{h(2h-5)}{h} = 2h-5$$

(b) $f(x) = \dfrac{1}{x^2+3}, \quad$ for $\quad a = 0 \quad \dfrac{f(0+h)-f(0)}{h} = -\dfrac{h}{3(h^2+3)}$

(4) For each function given in exercise (3), evaluate $\dfrac{f(x+h)-f(x)}{h}$ for any x in the domain of the function. Simplify to the extent possible.

(a) $\dfrac{f(x+h)-f(x)}{h} = 4x+2h-1$

(b) $\dfrac{f(x+h)-f(x)}{h} = \dfrac{\frac{1}{(x+h)^2+3}-\frac{1}{x^2+3}}{h} = \dfrac{(x^2+3)-((x+h)^2+3)}{h((x+h)^2+3)(x^2+3)}$

$$= \frac{x^2+3-(x^2+2xh+h^2+3)}{h((x+h)^2+3)(x^2+3)} = \frac{-2xh-h^2}{h((x+h)^2+3)(x^2+3)} =$$

$$= \frac{-h(2x+h)}{h((x+h)^2+3)(x^2+3)} = \frac{-(2x+h)}{((x+h)^2+3)(x^2+3)}$$

(5) Let $y = \log_a(M)$ so that $a^y = M$. Take the logarithm base b of both sides of the exponential equation, and using logarithm properties derive the change of base formula. (That is, show that $\log_b(M) = \frac{\log_a(M)}{\log_a(b)}$.) From $a^y = M$, using the power log property

$$\log_b(M) = \log_b(a^y) = y\log_b(a).$$

Solving for y, and recalling that $y = \log_a(M)$,

$$y\log_b(a) = \log_b(M) \implies y = \frac{\log_a(M)}{\log_a(b)} \quad \text{that is,} \quad \log_b(M) = \frac{\log_a(M)}{\log_a(b)}.$$

(6) Identify each statement as true or false. (Full disclosure, some of these statements are embarrassingly ludicrous.)

(a) $\dfrac{\ln(x)}{x} = \ln$ This is utter nonsense.

(b) $\log_4(x) = \dfrac{\log_5(x)}{\log_5(4)}$ True, this is an application of change of base.

(c) $(e^x)^2 = e^{2x}$ True by properties of exponents.

(d) $\ln x = \dfrac{1}{x}$ This is again some nonsense statement.

(e) $\log_a(x-y) = \dfrac{\log_a(x)}{\log_a(y)}$ False, this looks like an incorrect variation on a genuine property.

(f) $\log(8^9) = 9\log(8)$ True, this is the power property.

(g) $e^{9x} = 9e^x$ False. It is true that $e^{9x} = (e^x)^9$, but it's not $9e^x$.

(7) Each of the following functions is one to one on the indicated interval. Identify the inverse function.

(a) $f(x) = \dfrac{5x+3}{x-4}$ Let $y = f(x)$, then switch x and y and solve for the new y.

$$\begin{aligned} x &= \frac{5y+3}{y-4} \\ x(y-4) &= 5y+3 \\ xy-5y &= 4x+3 \\ y(x-5) &= 4x+3 \\ y &= \frac{4x+3}{x-5} \end{aligned}$$

$f^{-1}(x) = \dfrac{4x+3}{x-5}$.

(b) $g(x) = 3x^5 + 7$ $g^{-1}(x) = \sqrt[5]{\dfrac{x-7}{3}}$.

(c) $S(x) = e^{2x^3}$ $S^{-1}(x) = \sqrt[3]{\dfrac{\ln x}{2}}$.

(8) Use composition to show that the given functions are inverses.

$$f(x) = \sqrt[5]{\frac{x-1}{2x}} \quad \text{and} \quad f^{-1}(x) = \frac{1}{1-2x^5}$$

There are two compositions to consider, $(f \circ f^{-1})(x)$ and $(f^{-1} \circ f)(x)$. Here's one of those

$$(f^{-1} \circ f)(x) = f^{-1}(f(x)) = f^{-1}\left(\sqrt[5]{\frac{x-1}{2x}}\right) = \frac{1}{1-2\left(\sqrt[5]{\frac{x-1}{2x}}\right)^5} =$$

$$= \frac{1}{1-2\left(\frac{x-1}{2x}\right)} = \frac{1}{1-\frac{x-1}{x}} = \frac{1}{1-\left(1-\frac{1}{x}\right)} = \frac{1}{\frac{1}{x}} = x$$

So the function labeled as f^{-1} is definitely the inverse of f. The other composition should similarly result in $(f \circ f^{-1})(x) = x$.

(9) Evaluate each expression without a calculator

(a) $\log_3(1) = 0$ (b) $\log_2 \frac{1}{32} = -5$ (c) $\ln\sqrt{e} = \frac{1}{2}$

(d) $\log(0.0001) = -4$ (e) $\log_4(2^7) = \frac{7}{2}$ (f) $\log_\pi \pi = 1$

(10) Express as a single logarithm. Simplify if possible.

(a) $4\ln x + \frac{1}{3}\ln y - 2\ln z = \ln\left(\frac{x^4\sqrt[3]{y}}{z^2}\right)$ (b) $\log_2(x^3-8) - \log_2(x^2+2x+4)$

For (b), it helps to recall the difference of cubes $x^2 - 2^3 = (x-2)(x^2+2x+2^2)$.

(b) $\log_2(x^3-8) - \log_2(x^2+2x+4) = \log_2\left(\frac{x^3-8}{x^2+2x+4}\right) = \log_2\left(\frac{(x-2)(x^2+2x+4)}{x^2+2x+4}\right) = \log_2(x-2)$

(11) Expand as a sum or difference of logarithms.

(a) $\ln\sqrt[4]{wr^2}$ (b) $\log\sqrt[3]{\frac{M^2}{N}} = \frac{2}{3}\log|M| - \frac{1}{3}\log(N)$

(a) $\ln\sqrt[4]{wr^2} = \ln(wr^2)^{1/4} = \frac{1}{4}\ln(wr^2) = \frac{1}{4}(\ln w + \ln r^2) = \frac{1}{4}(\ln w + 2\ln|r|) = \frac{1}{4}\ln w + \frac{1}{2}\ln|r|$.

(12) Produce a plot of each function. Label any asymptotes and intercepts. See plots at the end.

(a) $y = e^{x-1}$ (b) $f(t) = \ln(-t)$

(c) $g(x) = e^x + 2$ (d) $y = \log_{1/2} x$

(13) Solve each equation. Obtain an exact solution.

(a) $\log_3(x) + \log_3(x+1) = \log_3(2) + \log_3(x+3)$

$$\log_3(x(x+1)) = \log_3(2(x+3)) \implies \log_3(x^2+x) = \log_3(2x+6)$$

$$x^2 + x = 2x + 6 \implies x^2 - x - 6 = 0 \implies (x-3)(x+2) = 0.$$

This suggests two solutions 3 and -2. Plugging each into the original equation shows that 3 does solve the equation. However, $\log_3(-2) + \log_3(-2+1)$ isn't defined, so -2 is NOT a solution. The only solution is 3. Contrast this with the results of problem (b).

(b) $\log_3(x^2+x) = \log_3(2) + \log_3(x+3)$ 3 and -2

(c) $e^x + e^{-x} = 3$ $\ln\left(\frac{3+\sqrt{5}}{2}\right)$ and $\ln\left(\frac{3-\sqrt{5}}{2}\right)$ (Hint: multiply both sides by e^x to get a quadratic equation in e^x.)

(d) $5^{x+1} = 3^{2x-1}$ $\dfrac{\ln 3 + \ln 5}{2\ln 3 - \ln 5}$. The natural log can be replaced with any other base.

(14) Given one trigonometric value of an acute angle, find the remaining five trigonometric values.

(a) $\cot\alpha = 3$

(b) $\sec\beta = \dfrac{7}{2}$

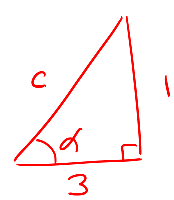
(c) $\sin\sigma = \dfrac{12}{13}$

See the end for (a) worked out.

(b) $\sin\beta = \dfrac{\sqrt{45}}{7}$, $\cos\beta = \dfrac{2}{7}$, $\tan\beta = \dfrac{\sqrt{45}}{2}$, $\cot\beta = \dfrac{2}{\sqrt{45}}$, $\csc\beta = \dfrac{7}{\sqrt{45}}$

(b) $\sec\sigma = \dfrac{13}{5}$, $\cos\sigma = \dfrac{5}{13}$, $\tan\sigma = \dfrac{12}{5}$, $\cot\sigma = \dfrac{5}{12}$, $\csc\sigma = \dfrac{13}{12}$

(a) $\cot\alpha = 3 = \dfrac{\text{adj}}{\text{opp}}$



a representative triangle with $\dfrac{\text{adj}}{\text{opp}} = 3$

$c^2 = 3^2 + 1^2 = 10$ $c = \sqrt{10}$

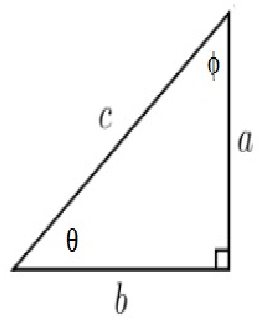
$\sin\alpha = \dfrac{1}{\sqrt{10}}$

$\cos\alpha = \dfrac{3}{\sqrt{10}}$

$\tan\alpha = \dfrac{1}{3}$

$\sec\alpha = \dfrac{\sqrt{10}}{3}$

$\csc\alpha = \sqrt{10}$



(15) The variables used in this problem are defined in the figure above Use the given information to solve for the remaining side lengths and indicated trigonometric values.

(i) $c = 6$ and $\sin\theta = \dfrac{2}{3}$. Find a, b, $\cos\theta$ and $\tan\theta$. $a = 4$, $b = 2\sqrt{5}$, $\cos\theta = \frac{\sqrt{5}}{3}$, and $\tan\theta = \frac{2}{\sqrt{5}}$

(ii) $a = 1$ and $\tan\phi = 5$. Find b, c, $\sin\theta$ and $\sin\phi$. $b = 5$, $c = \sqrt{26}$, $\sin\theta = \frac{1}{\sqrt{26}}$, and $\sin\phi = \frac{5}{\sqrt{26}}$

(iii) $b = 4$ and $\cos\phi = \dfrac{1}{\sqrt{5}}$. Find a, c, $\sin\phi$ and $\tan\phi$. $\cos\phi = \frac{a}{c} \implies c = \sqrt{5}a$. Since $a^2 + b^2 = c^2$,

$$a^2 + 4^2 = (\sqrt{5}a)^2 = 5a^2 \implies 4a^2 = 16 \implies a = 2$$

Thus $a = 2$ and $c = 2\sqrt{5}$ while $\sin\phi = \frac{4}{2\sqrt{5}} = \frac{2}{\sqrt{5}}$ and $\tan\phi = \frac{4}{2} = 2$.

(16) Evaluate each expression exactly without a calculator.

(a) $\sin 30° \cos 45° = \dfrac{1}{2\sqrt{2}}$

(b) $\csc 60° = \dfrac{2}{\sqrt{3}}$

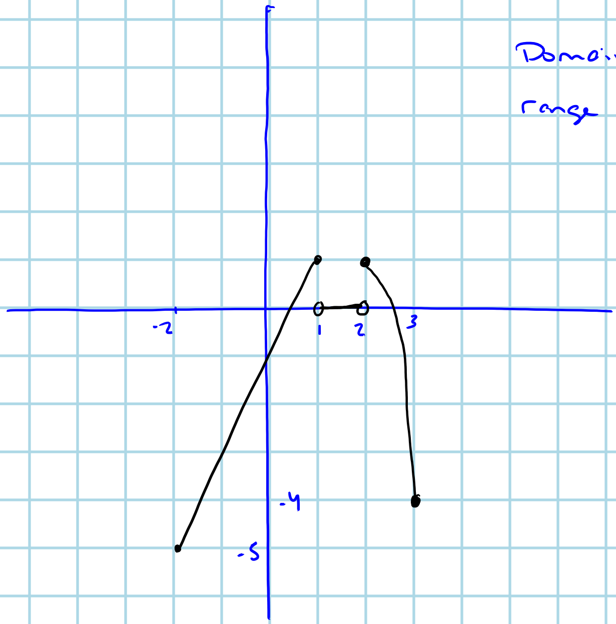
(c) $\sin 60° - 2\sin 30° \cos 30° = \frac{\sqrt{3}}{2} - 2\frac{1}{2}\frac{\sqrt{3}}{2} = 0$

(17) A regular pentagon is inscribed in a circle of radius 10. Find the perimeter of the pentagon.

Solutions below

(18) From a hot air balloon 2 km high, the angles of depression of two towns in line with the balloon and on the same side of the balloon are $81°$ and $13°$. How far apart are the towns (to the nearest km)?

2 (a)

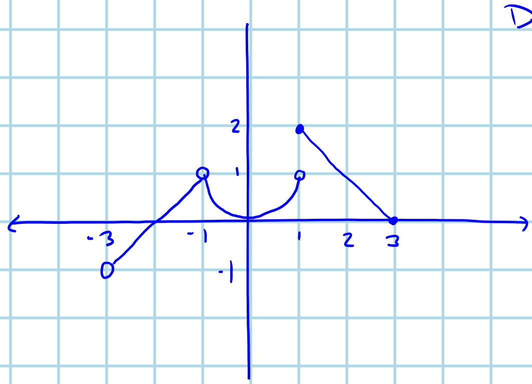


Domain $[-2, 3]$

range $[-5, 1]$

from the graph

(b)

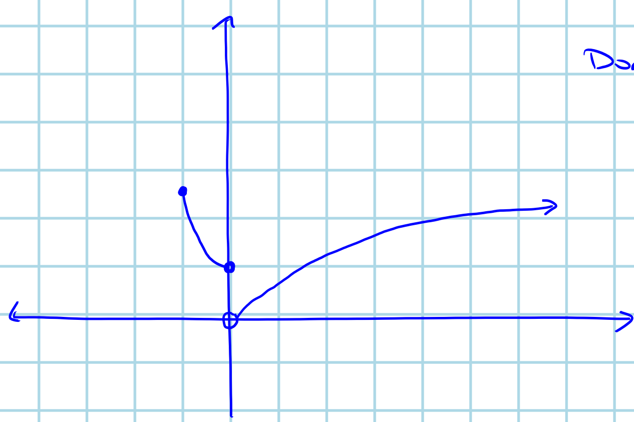


Domain $(-3, -1) \cup (-1, 3]$

range $(-1, 2]$

Seen from the graph

(c)

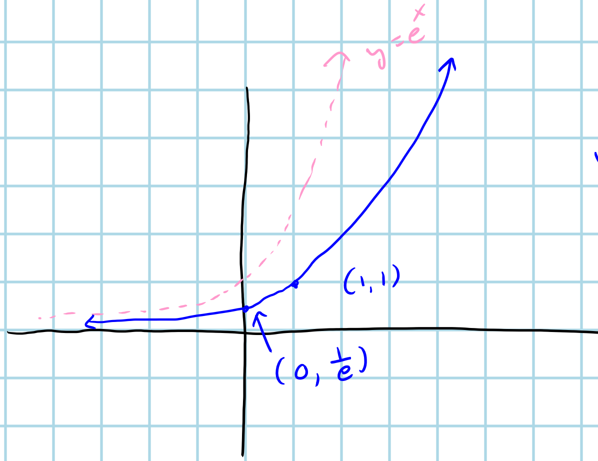


Domain $[-1, \infty)$

Range $(0, \infty)$

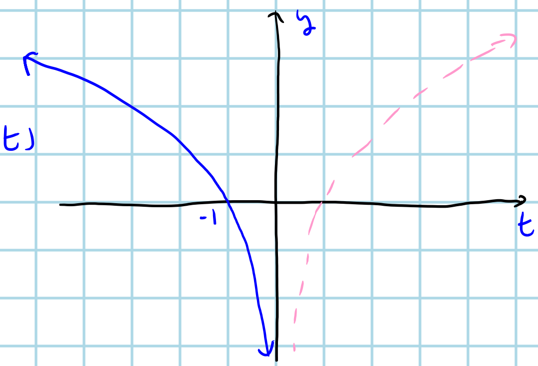
The range of $\ln x$ is $(-\infty, \infty)$, so every y value from zero to ∞ will be on this graph

12(a) $y = e^{x-1}$



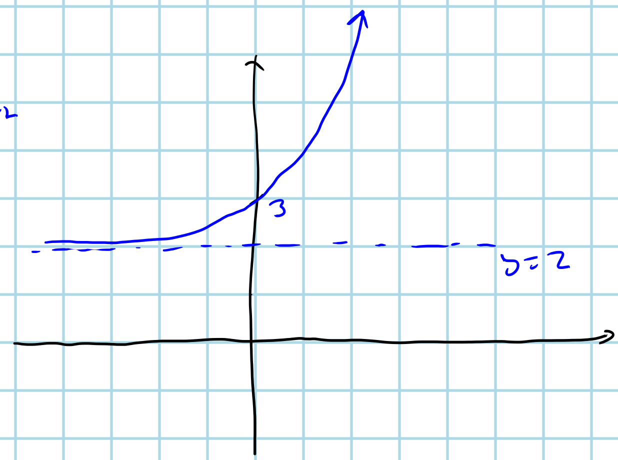
$y = e^x$ shifted 1 unit right

12(b) $f(t) = \ln(-t)$



$y = \ln t$ reflected in y-axis

12(c) $g(x) = e^{x+2}$



$y = e^x$ shifted up 2

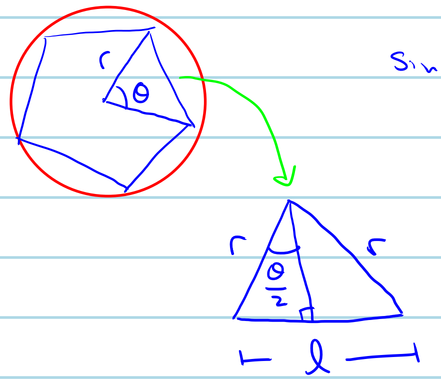
12(d) $y = \log_{\frac{1}{2}} x$

$\log_a(x)$ for $0 < a < 1$

17

The central angle $\theta = \frac{360^\circ}{5} = 72^\circ$

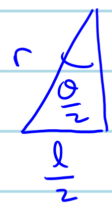
Since there are 5 of them



Let ℓ be the side length

we're given $r = 10$

we have a right triangle



$r = 10$, $\frac{\theta}{2} = \frac{72^\circ}{2} = 36^\circ$

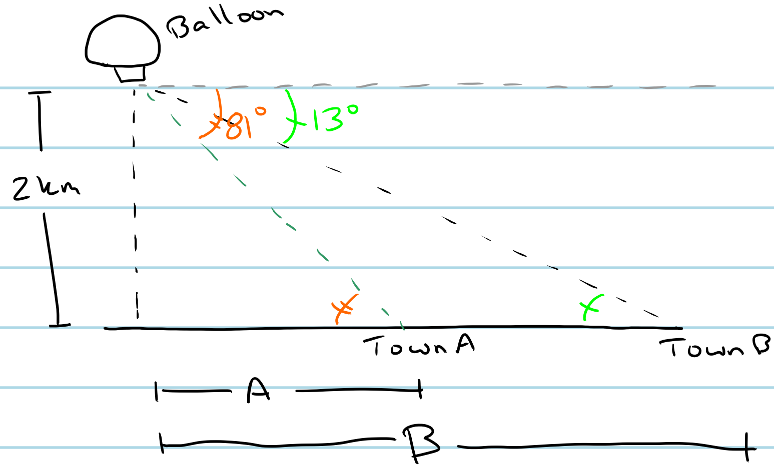
$\sin \frac{\theta}{2} = \frac{\frac{\ell}{2}}{r}$

So $\ell = 2r \sin \frac{\theta}{2} = 2 \cdot 10 \sin(36^\circ)$

There are 5 sides. The perimeter P satisfies

$P = 5\ell = 5 \cdot 2 \cdot 10 \sin 36^\circ \approx 58.8$ length units

18)



From the diagram

$$\tan 13° = \frac{2\text{km}}{B} \Rightarrow B = \frac{2}{\tan 13°}\ \text{km}$$

$$\tan 81° = \frac{2\text{km}}{A} \Rightarrow A = \frac{2}{\tan 81°}\ \text{km}$$

The distance between the towns d satisfies

$$d = B - A = \frac{2}{\tan 13°}\ \text{km} - \frac{2}{\tan 81°}\ \text{km}$$

$$\approx 8.35\ \text{km}$$

To the nearest kilometer, the distance is 8 km.