Solutions to Review for Exam 3

MATH 2306 (Ritter)

Sections Covered: 8, 9, 10, 11, 12, 13, 14

This review is provided as a courtesy to give some idea of what material is covered. Nothing else is intended or implied.

(1) Solve each IVP

(a) $y''-3y'+2y=0 \quad y(0)=0, \quad y'(0)=2 \qquad y=2e^{2x}-2e^{x}$

(b) $y''+2y'=0 \quad y(1)=0, \quad y'(1)=1 \qquad y=\frac{1}{2}-\frac{e^2}{2}e^{-2x}$

(c) $y''-2y'+5y=0 \quad y(0)=0, \quad y'(0)=2 \qquad y=e^{x}\sin(2x)$

(2) Find the general solution of each nonhomogeneous equation

(a) $y''+6y'+9y=e^{x}+3e^{-3x} \qquad y=c_1e^{-3x}+c_2xe^{-3x}+\frac{1}{16}e^{x}+\frac{3}{2}x^2e^{-3x}$

(b) $y''+y'-12y=2x \qquad y=c_1e^{-4x}+c_2e^{3x}-\frac{1}{6}x-\frac{1}{72}$

(c) $y''+y=4\cos x \qquad y=c_1\cos x+c_2\sin x+2x\sin x$

(3) Determine the **form** of the particular solution. (Do not bother trying to find any of the coefficients A, B, etc.)

(a) $y''-4y'+5y=x\cos 2x \qquad y_p=(Ax+B)\cos(2x)+(Cx+D)\sin(2x)$

(b) $y''+y=x^3+e^{x} \qquad y_p=Ax^3+Bx^2+Cx+D+Ee^{x}$

(c) $y''-4y'+5y=xe^{2x}\sin x \qquad y_p=(Ax^2+Bx)e^{2x}\sin x+(Cx^2+Dx)e^{2x}\cos x$

(d) $y''-2y'+y=1+e^{x} \qquad y_p=A+Bx^2e^{x}$

(4) For each homogeneous equation, write out the characteristic equation. If the equation doesn't have a characteristic equation, briefly state why.

(a) $3\dfrac{d^4y}{dx^4}-2\dfrac{d^3y}{dx^3}+\dfrac{dy}{dx}-4y=0$ $3m^4-2m^3+m-4=0$

(b) $4y''+2xy'+e^xy=0$ none exists, it's not constant coefficient

(c) $x^3y'''+2x^2y''-4xy'+y=0$ none exists, it's not constant coefficient

(d) $y^{(6)}+16y^{(4)}-12y''+y=0$ $m^6+16m^4-12m^2+1=0$

(5) Consider the nonhomogeneous equation $x^2y''+xy'-y=x$. In standard form, this equation is $y''+\frac{1}{x}y'-\frac{1}{x^2}y=x^{-1}$.

(a) Try using the method of undetermined coefficients to find a particular solution of the form $y_p=Ax+B$. It should fail.

(b) Maybe the form was wrong. Try it again with $y_p=Ax^{-1}$. More failure.

(c) Why is this approach failing? The left side is not constant coefficient. Of course, $g(x)=x^{-1}$ is also not a function type for which the method can be expected to work.

(d) It can be shown that $y_c=c_1x+c_2x^{-1}$. Use an appropriate method to find y_p. Using Variation of Parameters, $y_p=\frac{x}{2}\ln x$ is a particular solution. It's not clear that one would *guess* that the solution should have a logarithm factor. (What did I do with the $-\frac{1}{4}x$ part of y_p? Was that legit? It must have been, or I wouldn't be mentioning it ;).)

(6) A certain spring is 1 ft long with no mass attached. An object weighing 10 lbs is attached to the spring. The length of the spring with the mass attached is then 18 inches.

(a) Compute the mass m in slugs and the spring constant k in lbs/ft. $m=\frac{5}{16}$ slugs $k=20$ lb/ft

(b) If the object is initially at equilibirum and given a downward velocity of 1 ft/sec, find the displacement for $t > 0$. $x(t) = -\frac{1}{8}\sin(8t)$

(c) Next assume that a driving force of $f(t) = \cos(\gamma t)$ is applied to the object. What value of γ will result in pure resonance? Resonance frequency is the natural, circular frequency $\omega = 8$ /sec.

(d) Let $f(t) = \cos(3t)$. Determine the displacement for $t > 0$ assuming the object started from rest at equilibrium . $x(t) = -\frac{16}{275}\cos(8t) + \frac{16}{275}\cos(3t)$ This is the solution to the IVP

$$\frac{5}{16}x'' + 20x = \cos(3t), \quad x(0) = 0, \quad x'(0) = 0$$

(7) A 64 lb object is attached to a spring whose spring constant is 26 lb/ft. A dashpot provides damping that is numerically equal to 8 times the instantaneous velocity.

(a) Determine the mass m of the object in slugs. $m = 2$ slugs

(b) Assuming there is no external applied force, set up the differential equation for the displacement and determine if the motion is overdamped, underdamped or critically damped. $2x'' + 8x' + 26x = 0$, i.e. $x'' + 4x' + 13x = 0$. The system is underdamped with characteristic roots $r = -2 \pm 3i$.

(c) If the object is initially displaced 6 inches above equilibrium and given an initial upward velocity of 2 ft/sec, determine the displacement for $t > 0$. $x(t) = \frac{1}{2}e^{-2t}\cos(3t) + e^{-2t}\sin(3t)$

(8) A 200 volt battery is applied to a series circuit with inductance 2 henries, resistance 26 ohms and capacitance $\frac{1}{80}$ farads. Find the charge on the capacitor $q(t)$ for $t > 0$ assuming the initial charge and current are zero, $q(0) = 0$, $i(0) = 0$. $q(t) = \frac{25}{6}e^{-8t} - \frac{20}{3}e^{-5t} + \frac{5}{2}$

(9) Consider the nonhomogeneous equation $x^2y'' + xy' - 4y = 20x^3$.

(a) One solution of the associated homogeneous equation is $y_1 = x^2$. Find a second linearly independent one y_2. $y_2 = \frac{1}{x^2}$

(b) Find a particular solution y_p of the nonhomogeneous equation. $y_p = 4x^3$

(c) Solve the IVP: $x^2y'' + xy' - 4y = 20x^3,\ y(1) = 3,\ y'(1) = 6$. $y = -2x^2 + \frac{1}{x^2} + 4x^3$

(10) Use the method of variation of parameters to find a particular solution for each nonhomogeneous equation.

(a) $y''+y = \sec\theta\tan\theta \quad -\dfrac{\pi}{2} < \theta < \dfrac{\pi}{2}$ $\quad y_p = \theta\cos\theta + \sin\theta\ln(\sec\theta)$

$y_p = (\theta - \tan\theta)\cos\theta + \sin\theta\ln(\sec\theta)$ is also correct.

(b) $y''+3y'+2y = \sin(e^x)$ $\quad y_p = -e^{-2x}\sin(e^x)$

(11) Use the definition (i.e. compute an integral) to show that $\mathscr{L}\{e^{at}\} = \frac{1}{s-a}$ for $s > a$. Use the fact that $e^{-st}e^{at} = e^{-(s-a)t}$. The convergence of the integral will require $s - a > 0$.

(12) Compute the transform or inverse transform as indicated. (Use the table of Laplace transforms along with any necessary algebra or identities.)

(a) $\mathscr{L}\{(2t-3)^2\} = \mathscr{L}\{4t^2 - 12t + 9\} = \dfrac{8}{s^3} - \dfrac{12}{s^2} + \dfrac{9}{s}$

(b) $\mathscr{L}\{\cos^2 t - \sin^2 t\} = \mathscr{L}\{\cos(2t)\} = \dfrac{s}{s^2+4}$

(c) $\mathscr{L}\{e^{3t}+7\sin(2t)-t^5\} = \dfrac{1}{s-3} + \dfrac{14}{s^2+4} - \dfrac{120}{s^6}$

(d) $\mathscr{L}^{-1}\left\{\dfrac{1}{s^4} - \dfrac{s}{s^2+5}\right\} = \mathscr{L}^{-1}\left\{\dfrac{1}{6}\dfrac{3!}{s^4} - \dfrac{s}{s^2+(\sqrt{5})^2}\right\} = \dfrac{1}{6}t^3 - \cos(\sqrt{5}t)$

(e) $\mathscr{L}^{-1}\left\{\dfrac{5s+3}{s^2+s}\right\} = \mathscr{L}^{-1}\left\{\dfrac{3}{s} + \dfrac{2}{s+1}\right\} = 3 + 2e^{-t}$