Review for Exam III

MATH 2306 sec. 52

Sections Covered: 9, 10, 11, 12, 13, 14, 15

This review is provided as a courtesy to give some idea of what material is covered. Nothing else is intended or implied.

(1) Find the general solution of each nonhomogeneous equation

(a) $y''+6y'+9y=e^x+3e^{-3x}$ $\quad y=c_1e^{-3x}+c_2xe^{-3x}+\frac{1}{16}e^x+\frac{3}{2}x^2e^{-3x}$

(b) $y''+y'-12y=2x$ $\quad y=c_1e^{-4x}+c_2e^{3x}-\frac{1}{6}x-\frac{1}{72}$

(c) $y''+y=4\cos x$ $\quad y=c_1\cos x+c_2\sin x+2x\sin x$

(2) Determine the **form** of the particular solution. (Do not bother trying to find any of the coefficients A, B, etc.)

(a) $y''-4y'+5y=x\cos 2x$ $\quad y_p=(Ax+B)\cos(2x)+(Cx+D)\sin(2x)$

(b) $y''+y=x^3+e^x$ $\quad y_p=Ax^3+Bx^2+Cx+D+Ee^x$

(c) $y''-4y'+5y=xe^{2x}\sin x$ $\quad y_p=(Ax^2+Bx)e^{2x}\sin x+(Cx^2+Dx)e^{2x}\cos x$

(d) $y''-2y'+y=1+e^x$ $\quad y_p=A+Bx^2e^x$

(3) For each homogeneous equation, write out the characteristic equation. If the equation doesn't have a characteristic equation, briefly state why.

(a) $3\frac{d^4y}{dx^4}-2\frac{d^3y}{dx^3}+\frac{dy}{dx}-4y=0$ $\quad 3m^4-2m^3+m-4=0$

(b) $4y''+2xy'+e^xy=0$ none exists, it's not constant coefficient

(c) $x^3y'''+2x^2y''-4xy'+y=0$ none exists, it's not constant coefficient

(d) $y^{(6)}+16y^{(4)}-12y''+y=0$ $m^6+16m^4-12m^2+1=0$

(4) For each of the following nonhomogeneous equations, determine whether the method of undetermined coefficients **could** be used to determine y_p. If not, give a brief explanation. For each, assume that the complementary solution can be found.

(a) $3\frac{d^4y}{dx^4}-2\frac{d^3y}{dx^3}+\frac{dy}{dx}-4y=x^3e^x$ yes, it could

(b) $4y''+2y'+y=\frac{1}{1+x^2}$ nope, RHS is not of the correct type

(c) $x^3y'''+2x^2y''-4xy'+y=\sin(2x)+x$ nope, LHS is not constant coefficient

(d) $y^{(6)}+16y^{(4)}-12y''+y=x\ln x$ nope, RHS is not of the correct type

(5)A certain spring is 1 ft long with no mass attached. An object weighing 10 lbs is attached to the spring. The length of the spring with the mass attached is then 18 inches.

(a) Compute the mass m in slugs and the spring constant k in lbs/ft. $m=\frac{5}{16}$ slugs $k=20$ lb/ft

(b) If the object is initially at equilibirum and given a downward velocity of 1 ft/sec, find the displacement for $t>0$. $x(t)=-\frac{1}{8}\sin(8t)$

(c) Next assume that a driving force of $f(t)=\cos(\gamma t)$ is applied to the object. What value of γ will result in pure resonance? Resonance frequency is the natural, circular frequency $\omega=8$ /sec.

(6) A 64 lb object is attached to a spring whose spring constant is 26 lb/ft. A dashpot provides damping that is numerically equal to 8 times the instantaneous velocity.

(a) Determine the mass m of the object in slugs. $m=2$ slugs

(b) Assuming there is no external applied force, set up the differential equation for the displacement and determine if the motion is overdamped, underdamped or critically damped. $2x'' + 8x' + 26x = 0$, i.e. $x'' + 4x' + 13x = 0$. The system is underdamped with characteristic roots $r = -2 \pm 3i$.

(c) If the object is initially displaced 6 inches above equilibrium and given an initial upward velocity of 2 ft/sec, determine the displacement for $t > 0$. $x(t) = \frac{1}{2}e^{-2t}\cos(3t) + e^{-2t}\sin(3t)$

(7) A 200 volt battery is applied to a series circuit with inductance 2 henries, resistance 26 ohms and capacitance $\frac{1}{80}$ farads. Find the charge on the capacitor $q(t)$ for $t > 0$ assuming the initial charge and current are zero, $q(0) = 0$, $i(0) = 0$. $q(t) = \frac{25}{6}e^{-8t} - \frac{20}{3}e^{-5t} + \frac{5}{2}$

(8) Consider the nonhomogeneous equation $x^2y'' + xy' - 4y = 20x^3$.

(a) One solution of the associated homogeneous equation is $y_1 = x^2$. Find a second linearly independent one y_2. $y_2 = \frac{1}{x^2}$

(b) Find a particular solution y_p of the nonhomogeneous equation. $y_p = 4x^3$

(c) Solve the IVP: $x^2y'' + xy' - 4y = 20x^3,\ y(1) = 3,\ y'(1) = 6$. $y = -2x^2 + \frac{1}{x^2} + 4x^3$

(9) Find a particular solution for the nonhomogeneous equation.

$y''+2y+3y' = \sin(e^x)$ $\quad y_p = -e^{-2x}\sin(e^x)$

(10) Compute the transform or inverse transform as indicated. (Use the table of Laplace transforms along with any necessary algebra or identities.)

(a) $\mathscr{L}\{(2t-3)^2\} = \mathscr{L}\{4t^2 - 12t + 9\} = \dfrac{8}{s^3} - \dfrac{12}{s^2} + \dfrac{9}{s}$

(b) $\mathscr{L}\{\cos^2 t - \sin^2 t\} = \mathscr{L}\{\cos(2t)\} = \dfrac{s}{s^2+4}$

(c) $\mathscr{L}\{e^{3t}\sin(2t) + e^{-t}t^3\} = \dfrac{2}{(s-3)^2+4} + \dfrac{6}{(s+1)^4}$

(d) $\mathscr{L}^{-1}\left\{\dfrac{1}{s^4} - \dfrac{s}{s^2+5}\right\} = \mathscr{L}^{-1}\left\{\dfrac{1}{6}\dfrac{3!}{s^4} - \dfrac{s}{s^2+(\sqrt{5})^2}\right\} = \dfrac{1}{6}t^3 - \cos(\sqrt{5}t)$

(e) $\mathscr{L}^{-1}\left\{\dfrac{5s+3}{s^2+s}\right\} = \mathscr{L}^{-1}\left\{\dfrac{3}{s} - \dfrac{2}{s+1}\right\} = 3 + 2e^{-t}$

(f) $\mathscr{L}^{-1}\left\{\dfrac{s+8}{s^2+4s+13}\right\} = \mathscr{L}^{-1}\left\{\dfrac{s+2}{(s+2)^2+3^2} + 2\,\dfrac{3}{(s+2)^2+3^2}\right\} = e^{-2t}\cos(3t) + 2e^{-2t}\sin(3t)$

(11) Write each piecewise defined function in terms of appropriate unit step functions. Then find the Laplace transform of each.

(a) $f(t) = \begin{cases} t^2, & 0 \le t < 1 \\ e^t, & t \ge 1 \end{cases}$ $\quad f(t) = t^2 - t^2\mathscr{U}(t-1) + e^t\mathscr{U}(t-1)$

$$\mathscr{L}\{f(t)\} = \frac{2}{s^3} - e^{-s}\{(t+1)^2\} + e^{-s}\mathscr{L}\{e^{t+1}\} = \frac{2}{s^2} - e^{-s}\left(\frac{2}{s^3} + \frac{2}{s^2} + \frac{1}{s}\right) + e^{-s}\frac{e}{s-1}$$

(b) $f(t) = \begin{cases} 0, & 0 \le t < \frac{\pi}{4} \\ \cos(2t), & t \ge \frac{\pi}{4} \end{cases}$ $\quad f(t) = \cos(2t)\mathscr{U}\left(t - \dfrac{\pi}{4}\right)$

$$\mathscr{L}\{f(t)\} = e^{-\frac{\pi}{4}s}\mathscr{L}\left\{\cos\left(2\left(t + \frac{\pi}{4}\right)\right)\right\} = e^{-\frac{\pi}{4}s}\mathscr{L}\left\{\cos\left(2t + \frac{\pi}{2}\right)\right\} = e^{-\frac{\pi}{4}s}\mathscr{L}\left\{-\sin(2t)\right\} = \frac{-2e^{-\frac{\pi}{4}s}}{s^2+4}$$

(12) Find the inverse Laplace transform of each function.

(a) $\mathscr{L}^{-1}\left\{\dfrac{6e^{-2s}}{s^4} + \dfrac{e^{-s}}{s^2+1}\right\} = (t-2)^3\mathscr{U}(t-2) + \sin(t-1)\mathscr{U}(t-1)$

(b) $\mathscr{L}^{-1}\left\{\dfrac{(s-2)e^{-\pi s}}{s^2-s}\right\} = \mathscr{L}^{-1}\left\{\left(\dfrac{2}{s} - \dfrac{1}{s-1}\right)e^{-\pi s}\right\} = 2\mathscr{U}(t-\pi) - e^{t-\pi}\mathscr{U}(t-\pi)$