Solutions to Review for Exam III

MATH 2306 sections 51 & 54

Sections Covered: 4.6, 4.9, 5.1, 7.1, 7.2

This review is provided as a courtesy to give some idea of what material is covered. Nothing else is intended or implied.

(1) A 1 kg mass is attached to a spring whose spring constant is 13 N/m. The surrounding medium offers a resistance that is numerically equal to 6 times the instantaneous velocity. The mass is released from rest 2 m above the equilibrium position. Determine the equation of motion. The position $x(t)=-e^{-3t}(2\cos 2t+3\sin 2t)$.

(2) An LRC series circuit exhibits free electrical vibrations. If the inductance $L=1h$ and the capacitance $C=0.04f$, determine the resistance R ($R>0$) such that the electrical vibrations are **(a)** overdamped ($R>10$), **(b)** underdamped ($0\le R<10$), and **(c)** critically damped ($R=10$).

(3) Find the Laplace transform using the definition and specify its domain.

$$f(t)=te^{2t}\qquad \mathscr{L}\{f(t)\}=\frac{1}{(s-2)^2},\quad s>2$$

(4) Find the Laplace transform using any method.

(a) $f(t)=(t-1)^2-e^{-3t}\qquad \mathscr{L}\{f(t)\}=\dfrac{2}{s^3}-\dfrac{2}{s^2}+\dfrac{1}{s}+\dfrac{1}{s+3},\quad s>0$

(b) $f(t)=t+\sin\pi t\qquad \mathscr{L}\{f(t)\}=\dfrac{1}{s^2}+\dfrac{\pi}{s^2+\pi^2},\quad s>0$

(c) $f(t)=\begin{cases} t, & 0\le t<1\\ 1, & 1\le t\end{cases}\qquad \mathscr{L}\{f(t)\}=\dfrac{1}{s^2}-\dfrac{e^{-s}}{s^2},\quad s>0$

(5) Find the inverse Laplace transform using any method.

(a) $F(s)=\dfrac{1}{s^2-25}\qquad \mathscr{L}^{-1}\{F(s)\}=\dfrac{1}{10}e^{5t}-\dfrac{1}{10}e^{-5t}=\dfrac{1}{5}\sinh(5t)$

(b) $F(s) = \dfrac{2s+5}{s^3+3s}$ $\mathscr{L}^{-1}\{F(s)\} = \dfrac{5}{3} + \dfrac{2}{\sqrt{3}}\sin(\sqrt{3}t) - \dfrac{5}{3}\cos(\sqrt{3}t)$

(c) $F(s) = \dfrac{4}{s(s+1)}$ $\mathscr{L}^{-1}\{F(s)\} = 4 - 4e^{-t}$

(6) Solve the IVP using the Laplace transform.

(a) $y'' + 4y = 1 \quad y(0) = 0, \quad y'(0) = -1$ $y = \dfrac{1}{4} - \dfrac{1}{4}\cos(2t) - \dfrac{1}{2}\sin(2t)$

(b) $y'' - y = 2\cos(5t) \quad y(0) = 0, \quad y'(0) = 0$ $y = \dfrac{1}{26}e^{t} + \dfrac{1}{26}e^{-t} - \dfrac{1}{13}\cos(5t)$

(7) Find $\mathscr{L}\{f(t)\}$ given that

$$\mathscr{L}\{f'(t)\} = \ln\left(\frac{s^2+4}{s^2}\right) \quad \text{and} \quad f(0) = 1$$

$$\mathscr{L}\{f(t)\} = \frac{1}{s}\ln\left(\frac{s^2+4}{s^2}\right) + \frac{1}{s}$$

(8) Find the general solution of the ODE.

$y'' + y = \csc x$ $y = c_1\cos x + c_2\sin x - x\cos x + \sin x \ln|\sin x|$

(9) Find the general solution of the ODE for which one solution to the associated homogeneous equation, y_1, is given.

$$x^2y'' + 3xy' - 3y = 15x^2, \quad y_1(x) = x^{-3}$$

$$y = c_1x^{-3} + c_2x + 3x^2$$

(10) A 160 pound weight is attached to an industrial spring causing it to stretch from 9 feet to 10.28 feet. The weight is then driven from rest at equilibrium by an external for $F(t) = F_0\cos(\gamma t)$. At what frequency γ will the external force induce pure resonance? (Take $g = 32$ feet per second squared.)

$$k = \frac{160}{1.28}, \quad m = \frac{160}{32} \quad \Longrightarrow \quad \gamma_{res} = \sqrt{\frac{k}{m}} = 5$$

(11) Solve the IVP.

$$\begin{aligned} \tfrac{dx}{dt} &= y \qquad x(0) = 1 \\ \tfrac{dy}{dt} &= x \qquad y(0) = 0 \end{aligned}$$

Note that the hyperbolic sine and cosine provide a nice, compact formulation of the solution.

$$x(t) = \frac{1}{2}e^t + \frac{1}{2}e^{-t} = \cosh t, \quad y(t) = \frac{1}{2}e^t - \frac{1}{2}e^{-t} = \sinh t$$