Solutions to Practice for Exam 3 MATH 3260 Fall 2017

Sections Covered: 6.1, 6.2, 6.3, 6.4, 6.7, 5.1, 5.2

This practice exam is intended to give you a rough idea of the types of problems you can expect to encounter. **Nothing else is intended or implied.**

(1) Suppose A is a square matrix and 2 is an eigenvalue of A. Explain why the matrix $A - 2I$ is a singular matrix. $(A - 2I)\mathbf{x} = \mathbf{0}$ has a nontrivial solution $\mathbf{x}$. By the invertible matrix theorem, it follows that $A - 2I$ is singular.

(2) Determine the eigenvalues of the matrix $\begin{bmatrix} 1 & 2 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 2 \end{bmatrix}$. $\lambda_1 = 1, \lambda_2 = -1, \lambda_3 = 2$

(3) For each eigenvalue of the matrix in the previous problem find a basis for the corresponding eigenspace. For λ_1,$\{(1,0,0)\}$, for λ_2,$\{(1,-1,0)\}$, for λ_3,$\{(2,1,3)\}$.

(4) Determine all real or complex eigenvalues of the matrix $\begin{bmatrix} 7 & -5 \\ 1 & 3 \end{bmatrix}$. $\lambda = 5 \pm i$

(5) Find a unit vector in the direction of $\mathbf{v} = (1,2,1,-3)$. $\left(\frac{1}{\sqrt{15}}, \frac{2}{\sqrt{15}}, \frac{1}{\sqrt{15}}, -\frac{3}{\sqrt{15}}\right)$

(6) Write the vector $\mathbf{u} = (0,-2,3,2)$ in the form $\mathbf{u} = \hat{\mathbf{u}} + \mathbf{z}$ where $\hat{\mathbf{u}}$ is parallel to $\mathbf{v}$ and $\mathbf{z}$ is orthogonal to $\mathbf{v}$ for the vector $\mathbf{v}$ in problem (5). Use this to find the distance between the point $(0,-2,3,2)$ and the line Span$\{\mathbf{v}\}$ in $\mathbb{R}^4$. $\mathbf{u} = \left(-\frac{7}{15}, -\frac{14}{15}, -\frac{7}{15}, \frac{21}{15}\right) + \left(\frac{7}{15}, -\frac{16}{15}, \frac{52}{15}, \frac{9}{15}\right)$. This is $\hat{\mathbf{u}} + \mathbf{z}$ written from left to right. The distance is $\|\mathbf{z}\| = \sqrt{3090}/15$.

(7) Find a basis for the orthogonal complement $[\,\mathrm{Row}(A)]^{\perp}$ of the row space of the given matrix.

Use [Row$(A)]^{\perp}$ =Nul(A).

$$A = \begin{bmatrix} 5 & 1 & 2 & 2 & 0 \\ 3 & 3 & 2 & -1 & -12 \\ 8 & 4 & 4 & -5 & 12 \\ 2 & 1 & 1 & 0 & -2 \end{bmatrix} \quad \left\{ \begin{bmatrix} -\frac{1}{3} \\ -\frac{1}{3} \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{10}{3} \\ \frac{26}{3} \\ 0 \\ 4 \\ 1 \end{bmatrix} \right\}$$

(8) Find an orthonormal basis for the subspace of $\mathbb{R}^4$ spanned by the vectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 2 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 2 \\ 2 \\ 0 \\ 1 \end{bmatrix}, \quad \left\{ \begin{bmatrix} \frac{1}{\sqrt{6}} \\ 0 \\ \frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \end{bmatrix}, \begin{bmatrix} \frac{1}{\sqrt{12}} \\ -\frac{3}{\sqrt{12}} \\ \frac{1}{\sqrt{12}} \\ -\frac{1}{\sqrt{12}} \end{bmatrix}, \begin{bmatrix} \frac{7}{\sqrt{68}} \\ \frac{3}{\sqrt{68}} \\ -\frac{1}{\sqrt{68}} \\ -\frac{3}{\sqrt{68}} \end{bmatrix} \right\}$$

(9) Define and inner product on $\mathbb{R}^2$ by $\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{v}^T A \mathbf{u}$ where $A = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix}$. Relative to this inner product

(a) Find the norm of $(1,1)$. $\|(1,1)\| = 4$

(b) Find a unit vector in the direction of $(1,1)$. $\left(\frac{1}{4}, \frac{1}{4}\right)$

(c) Characterize the set of all vectors that are orthogonal to $(1,2)$.(x_1, x_2) such that $11x_1 + 13x_2 = 0$

(d) Find the distance between $(1,1)$ and $(1,2)$. $\|(1,1) - (1,2)\| = \sqrt{5}$

(10) Find the least squares best fit line to the data

x	0	1	2	3	4	5
y	-1	1	1	3	4	6

$y = \frac{46}{35}x - \frac{20}{21}$

(11) For the product $\langle \mathbf{p}, \mathbf{q} \rangle = \mathbf{p}(-1)\mathbf{q}(-1) + \mathbf{p}(0)\mathbf{q}(0) + \mathbf{p}(1)\mathbf{q}(1)$ on $\mathbb{P}_2$, show that for each $\mathbf{p}$

$$\langle \mathbf{p}, \mathbf{p} \rangle \geq 0, \quad \text{and} \quad \langle \mathbf{p}, \mathbf{p} \rangle = 0$$

if and only if $\mathbf{p} = \mathbf{0}$. If you set $\mathbf{p} = p_0 + p_1 t + p_2 t^2$, the conditions $\mathbf{p}(-1) = \mathbf{p}(0) = \mathbf{p}(1) = 0$ result in the equation $p_0 - p_1 + p_2 = 0$, $p_0 = 0$, and $p_0 + p_1 + p_2 = 0$. You can check that the only solution is $p_0 = p_1 = p_2 = 0$. Note that what is written here isn't an argument. It's just intended to let you know how you can construct an argument.