Practice for Exam 3 (Ritter) MATH 3260 Spring 2018

Sections Covered: 4.4, 4.5, 4.6, 6.1, 6.2, 6.3, 6.4, 5.1, 5.2

These practice problems are intended to give you a rough idea of the types of problems you can expect to encounter. **Nothing else is intended or implied.**

(1) For each matrix A, find bases for Nul$A$, Col$A$, and Row$A$. Determine both rank$A$ and dim(Nul$A$).

$$\text{(a)}\quad A = \begin{bmatrix} 1 & 3 & 4 & -1 & 2 \\ 2 & 6 & 6 & 0 & -3 \\ 3 & 9 & 3 & 6 & -3 \\ 3 & 9 & 0 & 9 & 0 \end{bmatrix}, \quad \text{(b)}\quad A = \begin{bmatrix} 1 & 1 & -2 & 0 & 1 & -2 \\ 1 & 2 & -3 & 0 & -2 & -3 \\ 1 & -1 & 0 & 0 & 1 & 6 \\ 1 & -2 & 2 & 1 & -3 & 0 \\ 1 & -2 & 1 & 0 & 2 & -1 \end{bmatrix}$$

All spanning sets shown are bases.

$$\text{(a)}\quad \text{Nul}A = \text{Span}\left\{ \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right\}, \quad \text{Col}A = \text{Span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 6 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ -3 \\ 0 \end{bmatrix} \right\}$$

$$\text{Row}A = \text{Span}\left\{ \begin{bmatrix} 1 \\ 3 \\ 0 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}, \quad \text{rank}A = 3, \quad \dim(\text{Nul}A) = 2$$

(b) $\text{Nul}A = \text{Span}\left\{\begin{bmatrix}-1\\-1\\-1\\1\\0\\0\end{bmatrix}\right\}$, $\text{Col}A = \text{Span}\left\{\begin{bmatrix}1\\1\\1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\2\\-1\\-2\\-2\end{bmatrix}, \begin{bmatrix}-2\\-3\\0\\2\\1\end{bmatrix}, \begin{bmatrix}1\\-2\\1\\-3\\2\end{bmatrix}, \begin{bmatrix}-2\\-3\\6\\0\\-1\end{bmatrix}\right\}$

$\text{Row}A = \text{Span}\left\{\begin{bmatrix}1\\0\\0\\1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\\1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\\1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\0\\0\\1\end{bmatrix}\right\}$, $\text{rank}A = 5$, $\dim(\text{Nul}A) = 1$

(2) Show that the given set is a basis for $\mathbb{R}^2$. Determine the change of coordinates matrix $P_{\mathcal{B}}$ and its inverse (use the order presented here). Then use this to find the indicated coordinate vectors.

$\mathcal{B} = \left\{\begin{bmatrix}5\\1\end{bmatrix}, \begin{bmatrix}2\\2\end{bmatrix}\right\}$, $P_{\mathcal{B}} = \begin{bmatrix}5 & 2\\1 & 2\end{bmatrix}$, $\det(P) = 8 \neq 0$ $P_{\mathcal{B}}^{-1} = \frac{1}{8}\begin{bmatrix}2 & -2\\-1 & 5\end{bmatrix}$

Determine $[\mathbf{x}]_{\mathcal{B}}$ for

(a) $\mathbf{x} = \begin{bmatrix}5\\1\end{bmatrix}$, $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix}1\\0\end{bmatrix}$ (b) $\mathbf{x} = \begin{bmatrix}2\\2\end{bmatrix}$, $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix}0\\1\end{bmatrix}$ (c) $\mathbf{x} = \begin{bmatrix}-2\\3\end{bmatrix}$ $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix}-\frac{5}{4}\\\frac{17}{8}\end{bmatrix}$

(3) Determine the dimension of the indicated vector space.

(a) The subspace of $\mathbb{R}^4$ of vectors whose components sum to zero. 3

(b) The subspace of $\mathbb{P}_4$ consisting of polynomials of the form $\mathbf{p}(t) = at^4 + b(t^2 - t)$, for real numbers a and b. 2

(c) The null space of a 5×8 matrix with a rank of 4. 4

(d) The row space of an $m \times n$ matrix whose null space has dimension 6. (It may be necessary to express the answer in terms of m or n or both.) $n - 6$

(4) Find a unit vector in the direction of $\mathbf{v} = (1, 2, 1, -3)$. $\left(\frac{1}{\sqrt{15}}, \frac{2}{\sqrt{15}}, \frac{1}{\sqrt{15}}, -\frac{3}{\sqrt{15}}\right)$

(5) Write the vector $\mathbf{u} = (0, -2, 3, 2)$ in the form $\mathbf{u} = \hat{\mathbf{u}} + \mathbf{z}$ where $\hat{\mathbf{u}}$ is parallel to $\mathbf{v}$ and $\mathbf{z}$ is orthogonal to $\mathbf{v}$ for the vector $\mathbf{v}$ in problem (4). Use this to find the distance between the point $(0, -2, 3, 2)$ and the line Span$\{\mathbf{v}\}$ in $\mathbb{R}^4$. $\mathbf{u} = \left(-\frac{7}{15}, -\frac{14}{15}, -\frac{7}{15}, \frac{21}{15}\right) + \left(\frac{7}{15}, -\frac{16}{15}, \frac{52}{15}, \frac{9}{15}\right)$. This is $\hat{\mathbf{u}} + \mathbf{z}$ written from left to right. The distance is $\|\mathbf{z}\| = \sqrt{3090}/15$.

(6) Find a basis for $[\ \text{Row}(A)]^{\perp}$, the orthogonal complement of the row space of the given matrix.

$$A = \begin{bmatrix} 5 & 1 & 2 & 2 & 0 \\ 3 & 3 & 2 & -1 & -12 \\ 8 & 4 & 4 & -5 & 12 \\ 2 & 1 & 1 & 0 & -2 \end{bmatrix} \quad \left\{ \begin{bmatrix} -\frac{1}{3} \\ -\frac{1}{3} \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{10}{3} \\ \frac{26}{3} \\ 0 \\ 4 \\ 1 \end{bmatrix} \right\}$$

(7) Find an orthonormal basis for the subspace of $\mathbb{R}^4$ spanned by the vectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 2 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 2 \\ 2 \\ 0 \\ 1 \end{bmatrix} \quad \left\{ \begin{bmatrix} \frac{1}{\sqrt{6}} \\ 0 \\ \frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \end{bmatrix}, \begin{bmatrix} \frac{1}{\sqrt{12}} \\ -\frac{3}{\sqrt{12}} \\ \frac{1}{\sqrt{12}} \\ -\frac{1}{\sqrt{12}} \end{bmatrix}, \begin{bmatrix} \frac{7}{\sqrt{68}} \\ \frac{3}{\sqrt{68}} \\ -\frac{1}{\sqrt{68}} \\ -\frac{3}{\sqrt{68}} \end{bmatrix} \right\}$$

(8) Suppose A is a square matrix and 2 is an eigenvalue of A. Explain why the matrix $A - 2I$ is a singular matrix. $(A - 2I)\mathbf{x} = \mathbf{0}$ has a nontrivial solution $\mathbf{x}$. By the invertible matrix theorem, it follows that $A - 2I$ is singular.

(9) Determine the eigenvalues of the matrix $\begin{bmatrix} 1 & 2 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 2 \end{bmatrix}$. $\lambda_1 = 1, \lambda_2 = -1, \lambda_3 = 2$

(10) For each eigenvalue of the matrix in the previous problem find a basis for the corresponing eigenspace. For λ_1,$\{(1,0,0)\}$, for λ_2,$\{(1,-1,0)\}$, for λ_3,$\{(2,1,3)\}$.