Practice for Exam 3 (Ritter) MATH 3260 Spring 2020

Sections Covered: 4.4, 4.5, 4.6, 6.1, 6.2, 6.3, 6.4
These practice problems are intended to give you a rough idea of the types of problems you can expect to encounter. **Nothing else is intended or implied.**

1. For each matrix A, find bases for Nul$A$, Col$A$, and Row$A$. Determine both rank$A$ and dim(Nul$A$).

$$\text{(a)}\quad A = \begin{bmatrix} 1 & 3 & 4 & -1 & 2 \\ 2 & 6 & 6 & 0 & -3 \\ 3 & 9 & 3 & 6 & -3 \\ 3 & 9 & 0 & 9 & 0 \end{bmatrix}, \quad \text{(b)}\quad A = \begin{bmatrix} 1 & 1 & -2 & 0 & 1 & -2 \\ 1 & 2 & -3 & 0 & -2 & -3 \\ 1 & -1 & 0 & 0 & 1 & 6 \\ 1 & -2 & 2 & 1 & -3 & 0 \\ 1 & -2 & 1 & 0 & 2 & -1 \end{bmatrix}$$

All spanning sets shown are bases.

$$\text{(a)}\quad \text{Nul}A = \text{Span}\left\{ \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right\}, \quad \text{Col}A = \text{Span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 6 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ -3 \\ 0 \end{bmatrix} \right\}$$

$$\text{Row}A = \text{Span}\left\{ \begin{bmatrix} 1 \\ 3 \\ 0 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}, \quad \text{rank}A = 3, \quad \dim(\text{Nul}A) = 2$$

$$\text{(b)}\quad \text{Nul}A = \text{Span}\left\{ \begin{bmatrix} -1 \\ -1 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}, \quad \text{Col}A = \text{Span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -1 \\ -2 \\ -2 \end{bmatrix}, \begin{bmatrix} -2 \\ -3 \\ 0 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 1 \\ -3 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ -3 \\ 6 \\ 0 \\ -1 \end{bmatrix} \right\}$$

$$\text{Row}A = \text{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}, \quad \text{rank}A = 5, \quad \dim(\text{Nul}A) = 1$$

2. Show that the given set is a basis for $\mathbb{R}^2$. Determine the change of coordinates matrix $P_{\mathcal{B}}$ and its inverse (use the order presented here). Then use this to find the indicated coordinate vectors.

$$\mathcal{B} = \left\{ \begin{bmatrix} 5 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \end{bmatrix} \right\}, \quad P_{\mathcal{B}} = \begin{bmatrix} 5 & 2 \\ 1 & 2 \end{bmatrix}, \quad \det(P) = 8 \neq 0 \quad P_{\mathcal{B}}^{-1} = \frac{1}{8}\begin{bmatrix} 2 & -2 \\ -1 & 5 \end{bmatrix}$$

Determine $[\mathbf{x}]_{\mathcal{B}}$ for

(a) $\mathbf{x} = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$, $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ (b) $\mathbf{x} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$, $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ (c) $\mathbf{x} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$ $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} -\frac{5}{4} \\ \frac{17}{8} \end{bmatrix}$

3. Determine the dimension of the indicated vector space.

(a) The subspace of $\mathbb{R}^4$ of vectors whose components sum to zero. 3

(b) The subspace of $\mathbb{P}_4$ consisting of polynomials of the form $\mathbf{p}(t) = at^4 + b(t^2 - t)$, for real numbers a and b. 2

(c) The null space of a 5×8 matrix with a rank of 4. 4

(d) The row space of an $m \times n$ matrix whose null space has dimension 6. (It may be necessary to express the answer in terms of m or n or both.) $n - 6$

(e) The subspace of $M^{3\times 3}$ consisting of matrices in which the entries in each column sum to zero (e.g. $[a_{ij}]$ such that $a_{1j} + a_{2j} + a_{3j} = 0$ for each $j = 1, 2, 3$.) 6

4. Find a unit vector in the direction of $\mathbf{v} = (1, 2, 1, -3)$. $\left(\frac{1}{\sqrt{15}}, \frac{2}{\sqrt{15}}, \frac{1}{\sqrt{15}}, -\frac{3}{\sqrt{15}}\right)$

5. Write the vector $\mathbf{u} = (0, -2, 3, 2)$ in the form $\mathbf{u} = \hat{\mathbf{u}} + \mathbf{z}$ where $\hat{\mathbf{u}}$ is parallel to $\mathbf{v}$ and $\mathbf{z}$ is orthogonal to $\mathbf{v}$ for the vector $\mathbf{v}$ in problem (4). Use this to find the distance between the point $(0, -2, 3, 2)$ and the line Span$\{\mathbf{v}\}$ in $\mathbb{R}^4$. $\mathbf{u} = \left(-\frac{7}{15}, -\frac{14}{15}, -\frac{7}{15}, \frac{21}{15}\right) + \left(\frac{7}{15}, -\frac{16}{15}, \frac{52}{15}, \frac{9}{15}\right)$. This is $\hat{\mathbf{u}} + \mathbf{z}$ written from left to right. The distance is $\|\mathbf{z}\| = \sqrt{3090}/15$.

6. Find a basis for $[\,\text{Row}(A)]^{\perp}$, the orthogonal complement of the row space of the given matrix.

$$A = \begin{bmatrix} 5 & 1 & 2 & 2 & 0 \\ 3 & 3 & 2 & -1 & -12 \\ 8 & 4 & 4 & -5 & 12 \\ 2 & 1 & 1 & 0 & -2 \end{bmatrix} \quad \left\{ \begin{bmatrix} -\frac{1}{3} \\ -\frac{1}{3} \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{10}{3} \\ \frac{26}{3} \\ 0 \\ 4 \\ 1 \end{bmatrix} \right\}$$

7. Find an orthonormal basis for the subspace of $\mathbb{R}^4$ spanned by the vectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 2 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 2 \\ 2 \\ 0 \\ 1 \end{bmatrix} \quad \left\{ \begin{bmatrix} \frac{1}{\sqrt{6}} \\ 0 \\ \frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \end{bmatrix}, \begin{bmatrix} \frac{1}{\sqrt{12}} \\ -\frac{3}{\sqrt{12}} \\ \frac{1}{\sqrt{12}} \\ -\frac{1}{\sqrt{12}} \end{bmatrix}, \begin{bmatrix} \frac{7}{\sqrt{68}} \\ \frac{3}{\sqrt{68}} \\ -\frac{1}{\sqrt{68}} \\ -\frac{3}{\sqrt{68}} \end{bmatrix} \right\}$$

8. Consider the set $\{\mathbf{u}_1, \mathbf{u}_2\} = \left\{ \begin{bmatrix} 2 \\ 6 \\ 0 \end{bmatrix}, \begin{bmatrix} -6 \\ 2 \\ 0 \end{bmatrix} \right\}$. Find the orthogonal projection of $\mathbf{y} = \begin{bmatrix} 3 \\ 5 \\ -4 \end{bmatrix}$ onto Span$\{\mathbf{u}_1, \mathbf{u}_2\}$, and find the distance between $\mathbf{y}$ and the plane Span$\{\mathbf{u}_1, \mathbf{u}_2\}$.

Calling the projection $\hat{\mathbf{y}}$, we have $\hat{\mathbf{y}} = \frac{\mathbf{y} \cdot \mathbf{u}_1}{\|\mathbf{u}_1\|^2}\mathbf{u}_1 + \frac{\mathbf{y} \cdot \mathbf{u}_2}{\|\mathbf{u}_2\|^2}\mathbf{u}_2 = \begin{bmatrix} 3 \\ 5 \\ 0 \end{bmatrix}$, and the distance is $\|\mathbf{y} - \hat{\mathbf{y}}\| = 4$.

9. The first four Chebyshev polynomials are

$$T_0(t) = 1, \quad T_1(t) = t, \quad T_2(t) = 2t^2 - 1, \quad T_3(t) = 4t^3 - 3t.$$

(a) Show that the set $\{T_0, T_1, T_2, T_3\}$ is linearly independent in $\mathbb{P}^3$. A quick way to do this is to use coordinate vectors relative to the standard basis $\mathcal{E} = \{1, t, t^2, t^3\}$. The coordinate vectors would be

$$[\mathbf{T}_0]_{\mathcal{E}} = (1,0,0,0), \quad [\mathbf{T}_1]_{\mathcal{E}} = (0,1,0,0), \quad [\mathbf{T}_2]_{\mathcal{E}} = (-1,0,2,0), \quad [\mathbf{T}_3]_{\mathcal{E}} = (0,-3,0,4)$$

Popping these into a matrix, and taking the determinant

$$\det([[\mathbf{T}_0]_{\mathcal{E}}\ [\mathbf{T}_1]_{\mathcal{E}}\ [\mathbf{T}_2]_{\mathcal{E}}\ [\mathbf{T}_3]_{\mathcal{E}}\ ]) = \det\left(\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}\right) = 8 \neq 0$$

The coordinate vectors are linearly independent, so the original vectors in $\mathbb{P}_3$ are linearly independent.

(b) Let $\mathbf{p}(t) = 1 + t + t^2 + t^3$. Find the coordinate vector $[\mathbf{p}]_{\mathcal{C}}$ relative to the basis $\mathcal{C} = \{T_0, T_1, T_2, T_3\}$. $[\mathbf{p}]_{\mathcal{C}} = \begin{bmatrix} 3/2 \\ 7/4 \\ 1/2 \\ 1/4 \end{bmatrix}$

(c) Find the polynomial $\mathbf{q}$ in $\mathbb{P}_3$ whose coordinate vector $[\mathbf{q}]_{\mathcal{C}} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$.$\mathbf{q}(t) = -2t + 2t^2 + 4t^3$

10. Suppose a nonhomogeneous system of six linear equations in eight unknowns has a solution, with two free variables. Is it possible to change some constants on the equations' right sides to make the new system inconsistent? (Why/why not?) No, it's always consistent. Here are some hints as to how this is known. See if you can piece together an explanation.

- Think about the dimensions of the matrix if you wrote the system in the form $A\mathbf{x} = \mathbf{b}$.
- Knowing the size of A (some $m \times n$), as yourself what the maximum rank could be and by virtue what the minimum possible dimension of the null space could be.
- We know that the number of free variables is 2. This relates to the dimension of the null space (how?).
- The rank tells you how many linearly independent columns you have. Can you argue that you have enough to span $\mathbb{R}^m$ (for the value of m that you already determined)?

11. Let $W = \text{Span}\left\{\begin{bmatrix}1\\0\\1\\2\end{bmatrix}, \begin{bmatrix}1\\-1\\1\\1\end{bmatrix}\right\}$. Find two nonzero, non-parallel vectors in $W^{\perp}$. Probably the easiest thing to do is to set up a matrix that has W as it's row space. Then you can use that $W^{\perp}$ is the null space of that matrix. The most obvious answer (not the only one) would be the pair $\left\{\begin{bmatrix}-1\\0\\1\\0\end{bmatrix}, \begin{bmatrix}-2\\-1\\0\\1\end{bmatrix}\right\}$. (This is the basis you get for the null space directly from the rref of the corresponding matrix.)