Review for Exam IV

MATH 1112 Section 54 Spring 2019

Sections Covered: 7.4, 7.1, 7.2, 7.3, 7.5,

Calculator Policy: Calculator use may be allowed on part of the exam. When instructions call for an **exact** solution, that indicates that a decimal approximation will not be accepted.

Problems marked with a blue asterisk * have detailed solutions.

(1) Consider each trigonometric equation. Determine if the equation is an identity or if it is conditional (i.e. not an identity). If it is an identity, prove it. If it is not an identity, find at least one value of the variable for which the equation is not satisfied.

(a) $\dfrac{\sin x}{1-\cos x} = \dfrac{1+\cos x}{\sin x}$ * An identity, see argument below

(b) $(\cos\theta + \sin\theta)^2 = 1$ Not an identity. Try evaluating the left at $\frac{\pi}{4}$. The left side is 2.

(c) $\sin^2\beta\tan^2\beta = \tan^2\beta - \sin^2\beta$ An identity. Produce your own argument.

(d) $\dfrac{\tan x + \cot x}{\csc x} = \sec x$ An identity. Produce your own argument.

(e) $\cos^2 t + \cot^2 t = \sin t \sec t$ Not an identity. Try evaluating the left at $\frac{\pi}{4}$. The left side is $\frac{3}{2}$ while the right side is 1.

4(a)
$$\begin{aligned}\frac{\sin x}{1-\cos x} &= \frac{\sin x}{1-\cos x}\left(\frac{1+\cos x}{1+\cos x}\right)\\ &= \frac{\sin x(1+\cos x)}{1-\cos^2 x}\\ &= \frac{\sin x(1+\cos x)}{\sin^2 x}\\ &= \frac{1+\cos x}{\sin x}\end{aligned}$$

(2) Given $\cos(u-v) = \cos u\cos v + \sin u \sin v$ and $\sin(u-v) = \sin u\cos v - \sin v\cos u$, obtain expressions for each of the following.

(a) $\cos(u+v)$ *

$$\begin{aligned}\cos(u+v) &= \cos(u-(-v))\\ &= \cos u\cos(-v)+\sin u\sin(-v)\\ &= \cos u\cos v+\sin u(-\sin v) \quad \text{(by symmetry)}\\ &= \cos u\cos v-\sin u\sin v\end{aligned}$$

(b) $\sin(u+v) = \sin u\cos v + \sin v\cos u$ You should provide details showing how this follows from the given. One way to get started is to use

$$\sin(u+v) = \cos\left(\frac{\pi}{2}-(u+v)\right) = \cos\left(\left(\frac{\pi}{2}-u\right)-v\right)$$

Now apply the given difference of angles formula for the cosine along with the cofunction identities.

(c) $\cos(2u) = \cos^2 u - \sin^2 u$ There are three equivlant expressions depending on your approach.

(d) $\sin(2u) = 2\sin u\cos u$ Again, provide some details to show how this follows from (b).

(3) Suppose $\sin\alpha = \frac{1}{3}$ and $\cos\beta = -\frac{1}{5}$. Futher suppose that $\frac{\pi}{2} < \alpha < \pi$ and $\pi < \beta < \frac{3\pi}{2}$. Evaluate each of the following exactly.

(a) $\cos(\alpha-\beta)$ * See details on the last pages.

(b) $\sin(2\beta) = \dfrac{2\sqrt{24}}{25}$

(c) $\sin(\alpha+\beta) = \dfrac{8\sqrt{3}-1}{15}$

(4) Find an equivalent algebraic expression for each of the following (*algebraic* meaning without any trigonometric functions).

(a) $\sin\left(\sin^{-1}(2x)\right) = 2x$

(b) $\cos\left(\sin^{-1}(2x)\right)$ * See last pages.

(c) $\sec\left(\tan^{-1} u\right) = \sqrt{u^2+1}$

(5) Use the sum of angles formula for the cosine that you obtained in (2)(a) to show that if $f(x) = \cos x$, then * See last pages.

$$\frac{f(x+h)-f(x)}{h} = \cos x\left(\frac{\cos h - 1}{h}\right) - \sin x\left(\frac{\sin h}{h}\right)$$

(6) Evaluate each expression exactly.

(a) $\cos(\ln(1)) = \cos(0) = 1$ (b) $\tan^{-1}(e^0) = \tan^{-1} 1 = \frac{\pi}{4}$ (c) $e^{\tan 0^\circ} = e^0 = 1$

(d) $e^{\cos 90^\circ} = e^0 = 1$ (e) $\sin^{-1}(\ln e) = \sin^{-1} 1 = \frac{\pi}{2}$ (f) $\cos^{-1}(\ln 1) = \cos^{-1} 0 = \frac{\pi}{2}$

(7) Find all solutions of the given trigonometric equation on the interval $[0, 2\pi)$.

(a) $2\sin x - \sqrt{3} = 0$ $\left\{\frac{\pi}{3}, \frac{2\pi}{3}\right\}$

(b) $\sin(2x) + \cos x = 0$ $\left\{\frac{\pi}{2}, \frac{7\pi}{6}, \frac{3\pi}{2}, \frac{11\pi}{6}\right\}$

(c) $\tan^2 x = 1$ $\left\{\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}\right\}$

(d) $2\cos^2 x - 1 = \cos x$ $\left\{0, \frac{2\pi}{3}, \frac{4\pi}{3}\right\}$

(e) $\sin x + 1 = 2\cos^2 x$ * See last pages.

(8) Evaluate each expression exactly. No calculator is needed for these.

(a) $\cos(75^\circ) = \dfrac{\sqrt{3}-1}{2\sqrt{2}}$

(b) $\cos(80^\circ)\cos(20^\circ) + \sin(80^\circ)\sin(20^\circ) = \dfrac{1}{2}$

(c) $\sin\left(\frac{\pi}{12}\right) = \sqrt{\dfrac{2-\sqrt{3}}{4}}$

(d) $\sin(12^\circ)\cos(57^\circ) - \sin(57^\circ)\cos(12^\circ) = -\dfrac{1}{\sqrt{2}}$

(e) $\cos\left(\frac{7\pi}{12}\right) = -\sqrt{\dfrac{2-\sqrt{3}}{4}}$

(f) $\tan\left(\frac{11\pi}{12}\right)$ $* = \tan\left(\dfrac{\pi}{4} + \dfrac{2\pi}{3}\right) = \dfrac{\tan\frac{\pi}{4} + \tan\frac{2\pi}{3}}{1 - \tan\frac{\pi}{4}\tan\frac{2\pi}{3}} = \dfrac{1-\sqrt{3}}{1+\sqrt{3}}$

(9) Solve each triangle given the information provided. (We use the standard convention that the angles A, B, C are opposite sides of length a, b, and c, respectively. Express angles in degrees, and round all values to the nearest hundredths.

(a) $C = 140°$, $b = 1$, $c = 9$ $B = 4.10°$, $A = 35.90°$, $a = 8.21$

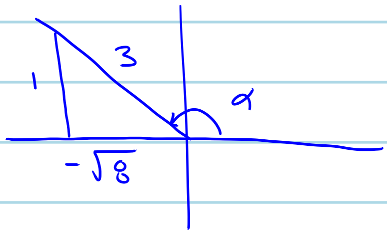
(b) $A = 35°$, $a = 6$, $b = 8$ There are two triangles. $B = 49.89°$, $C = 95.11°$, $c = 10.42$, or $B = 130.11°$, $C = 14.89°$, $c = 2.69$

(c) $a = 2$, $b = 3$, $C = 60°$ $c = 2.65$, $A = 40.89°$, $B = 79.11°$

(d) $a = 4$, $b = 3$, $c = 6$ $A = 36.34°$, $B = 26.38°$, $C = 117.28°$

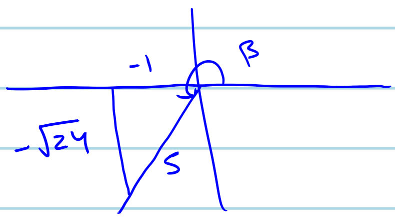
3 a) Draw representative triangles

$\sin\alpha = \frac{1}{3}$ $\quad \pi/2 < \alpha < \pi$ $\quad \alpha$ in Quad II



From the diagram $\cos\alpha = \frac{-\sqrt{8}}{3}$

$\cos\beta = \frac{-1}{5}$ $\quad \pi < \beta < \frac{3\pi}{2}$ $\quad \beta$ in Quad III



From this diagram
$\sin\beta = \frac{-\sqrt{24}}{5}$

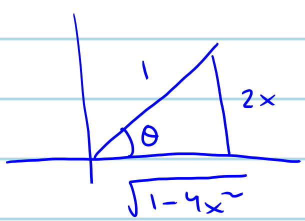
$$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

$$= \frac{-\sqrt{8}}{3}\left(\frac{-1}{5}\right) + \frac{1}{3}\left(\frac{-\sqrt{24}}{5}\right) = \frac{\sqrt{8}-\sqrt{24}}{15}$$

4 (b) $\cos(\sin^{-1}(2x))$ $\quad$ Let $\theta = \sin^{-1}(2x)$

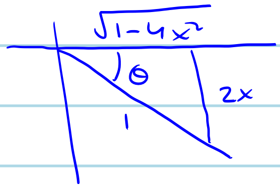
then $\sin\theta = 2x$ and $-\frac{\pi}{2} \le \theta \le \pi/2$

θ is in quad I or IV $\quad$ we can diagram both cases



(if $x > 0$)

or



(if $x < 0$)

In either case $\cos(\sin^{-1}(2x)) = \frac{\sqrt{1-4x^2}}{1} = \sqrt{1-4x^2}$

5) $f(x) = \cos x$ so $f(x+h) = \cos(x+h) = \cos x \cos h - \sin x \sin h$

Then

$$\frac{f(x+h) - f(x)}{h} = \frac{\cos(x+h) - \cos x}{h}$$

$$= \frac{\cos x \cos h - \sin x \sin h - \cos x}{h}$$

Collect $\cos x$ terms

$$= \frac{\cos x \cos h - \cos x - \sin x \sin h}{h}$$

$$= \frac{\cos x(\cos h - 1) - \sin x \sin h}{h}$$

$$= \frac{\cos x(\cos h - 1)}{h} - \frac{\sin x \sin h}{h}$$

$$= \cos x \left(\frac{\cos h - 1}{h}\right) - \sin x \left(\frac{\sin h}{h}\right)$$

7e) $\sin x + 1 = 2\cos^2 x$

Use $\cos^2 x = 1 - \sin^2 x$

$\sin x + 1 = 2(1 - \sin^2 x)$

$\sin x + 1 = 2 - 2\sin^2 x$

$\Rightarrow 2\sin^2 x + \sin x + 1 - 2 = 0$

$2\sin^2 x + \sin x - 1 = 0$

factor $(2\sin x - 1)(\sin x + 1) = 0$

Then $2\sin x - 1 = 0$ or $\sin x + 1 = 0$

$2\sin x - 1 = 0 \Rightarrow \sin x = \frac{1}{2}$

There are 2 solutions in $[0, 2\pi)$ one in each of quad I and II. These are

$\pi/6$ and $5\pi/6$

$\sin x + 1 = 0 \Rightarrow \sin x = -1$

There is one solution in $[0, 2\pi)$, namely $3\pi/2$.

The solution set is $\left\{ \frac{\pi}{6}, \frac{5\pi}{6}, \frac{3\pi}{2} \right\}$.