Solutions to Review for Exam IV

MATH 1113 sections 51 & 52 Fall 2018

Sections Covered: 6.3, 6.4, 6.5, 6.6, 7.4, 7.1, 7.2, 7.3, 7.5

Calculator Policy: Calculator use may be allowed on part of the exam. When instructions call for an **exact** solution, that indicates that a decimal approximation will not be accepted.

This review is provided as a courtesy to give some idea of what material is covered. Nothing else is intended or implied.

(1) Evaluate each trigonometric expression exactly if it exists. (Check with a calculator, but be able to do this without one. You can be sure I will ask you to do so on an exam.)

(a) $\cos\left(\frac{3\pi}{2}\right) = 0$ (b) $\cot(2\pi)$ undefined (c) $\csc\left(\frac{5\pi}{6}\right) = 2$

(d) $\sin\left(\frac{11\pi}{6}\right) = -\frac{1}{2}$ (e) $\tan\left(\frac{3\pi}{4}\right) = -1$ (f) $\cos\left(\frac{5\pi}{4}\right) = -\frac{1}{\sqrt{2}}$

(g) $\sec\left(\frac{5\pi}{2}\right)$ undefined (h) $\sec\left(\frac{2\pi}{3}\right) = -2$ (i) $\tan\left(\frac{5\pi}{3}\right) = -\sqrt{3}$

(2) State the domain and range of each of the six trigonometric functions. Use interval notation or set builder notation.

Function	Domain	Range
$y = \sin x$	$(-\infty, \infty)$	$[-1, 1]$
$y = \cos x$	$(-\infty, \infty)$	$[-1, 1]$
$y = \tan x$	$\left\{x : x \neq \frac{\pi}{2} + k\pi, \text{ for integer } k\right\}$	$(-\infty, \infty)$
$y = \csc x$	$\{x : x \neq k\pi \text{ for integer } k\}$	$(-\infty, -1] \cup [1, \infty)$
$y = \sec x$	$\left\{x : x \neq \frac{\pi}{2} + k\pi, \text{for integer } k\right\}$	$(-\infty, -1] \cup [1, \infty)$
$y = \cot x$	$\{x : x \neq k\pi \text{ for integer } k\}$	$(-\infty, \infty)$

(3) Identify the amplitude and period of each of each function.

(a) $f(x) = -3\cos\left(\frac{x}{2}\right) - 2$ (b) $g(x) = 4 - 4\sin\left(\pi x + \frac{\pi}{6}\right)$ (c) $F(x) = 4\sin\left(\frac{\pi}{4} - 2x\right)$

For $y = a\cos(bx - c) + d$ (or the version with a sine function), amplitude is $|a|$, and period is $\frac{2\pi}{|b|}$. So (a) has amplitude 3 and period 4π; (b) has amplitude 4 and period 2; (c) has amplitude 4 and period π.

(4) Consider each trigonometric equation. Determine if the equation is an identity or if it is conditional (i.e. not an identity). If it is an identity, prove it. If it is not an identity, find at least one value of the variable for which the equation is not satisfied.

(a) $\dfrac{\sin x}{1-\cos x} = \dfrac{1+\cos x}{\sin x}$ An identity, see argument below

(b) $(\cos\theta + \sin\theta)^2 = 1$ Not an identity. Try evaluating the left at $\frac{\pi}{4}$. The left side is 2.

(c) $\sin^2\beta\tan^2\beta = \tan^2\beta - \sin^2\beta$ An identity. Produce your own argument.

(d) $\dfrac{\tan x + \cot x}{\csc x} = \sec x$ An identity. Produce your own argument.

(e) $\cos^2 t + \cot^2 t = \sin t \sec t$ Not an identity. Try evaluating the left at $\frac{\pi}{4}$. The left side is $\frac{3}{2}$ while the right side is 1.

$$\text{4(a)} \quad \begin{aligned} \frac{\sin x}{1-\cos x} &= \frac{\sin x}{1-\cos x}\left(\frac{1+\cos x}{1+\cos x}\right) \\ &= \frac{\sin x(1+\cos x)}{1-\cos^2 x} \\ &= \frac{\sin x(1+\cos x)}{\sin^2 x} \\ &= \frac{1+\cos x}{\sin x} \end{aligned}$$

(5) Given $\cos(u-v) = \cos u\cos v + \sin u \sin v$ and $\sin(u-v) = \sin u\cos v - \sin v\cos u$, obtain expressions for each of the following.

(a) $\cos(u+v) = \cos u\cos v - \sin u\sin v$ Here are some details.

$$\begin{aligned} \cos(u+v) &= \cos(u-(-v)) \\ &= \cos u\cos(-v) + \sin u\sin(-v) \\ &= \cos u\cos v + \sin u(-\sin v) \quad \text{(by symmetry)} \\ &= \cos u\cos v - \sin u\sin v \end{aligned}$$

(b) $\sin(u+v) = \sin u\cos v + \sin v\cos u$ You should provide details showing how this follows from the given.

(c) $\cos(2u) = \cos^2 u - \sin^2 u$ There are three equivlant expressions depending on your approach.

(d) $\sin(2u) = 2\sin u \cos u$ Again, provide some details to show how this follows from (b).

(6) Suppose $\sin\alpha = \frac{1}{3}$ and $\cos\beta = -\frac{1}{5}$. Futher suppose that $\frac{\pi}{2} < \alpha < \pi$ and $\pi < \beta < \frac{3\pi}{2}$. Evaluate each of the following exactly.

(a) $\cos(\alpha - \beta) = \dfrac{\sqrt{8} - \sqrt{24}}{15}$ See details at the end after the word problems.

(b) $\sin(2\beta) = \dfrac{2\sqrt{24}}{25}$

(c) $\sin(\alpha + \beta) = \dfrac{8\sqrt{3} - 1}{15}$

(7) State the domain and the range of each of $f(x) = \sin^{-1}(x)$, $g(x) = \cos^{-1}(x)$ and $H(x) = \tan^{-1}(x)$ using interval notation.

Function	Domain	Range
$y = \sin^{-1} x$	$[-1, 1]$	$[-\frac{\pi}{2}, \frac{\pi}{2}]$
$y = \cos^{-1} x$	$[-1, 1]$	$[0, \pi]$
$y = \tan^{-1} x$	$(-\infty, \infty)$	$(-\frac{\pi}{2}, \frac{\pi}{2})$

* There was a typo! The range of the inverse cosine is $0 \le y \le \pi$ i.e. $[0, \pi]$ with square brackets!

(8) Evaluate each expression exactly if it exists. If it doesn't exist, state why.

(a) $\sin\left(\sin^{-1} 0.02\right) = 0.02$

(b) $\sin^{-1}(\sin 0.02) = 0.02$

(c) $\sin^{-1}[\sin(\pi)] = 0$ $\sin\pi = 0$ and $\sin^{-1} 0 = 0$

(d) $\cos^{-1}\left[\cos\left(-\frac{\pi}{4}\right)\right] = \frac{\pi}{4}$, $\cos\left(-\frac{pi}{4}\right) = \frac{1}{\sqrt{2}}$ and $\cos^{-1}\frac{1}{\sqrt{2}} = \frac{\pi}{4}$

(e) $\cos\left(\tan^{-1} 4\right) = \frac{1}{\sqrt{17}}$ Draw a triangle in quadrant I with opposite leg 4 and adjacent leg 1.

(f) $\csc\left[\cos^{-1}\left(\frac{2}{3}\right)\right] = \frac{3}{\sqrt{5}}$ Draw a triangle in quadrant I with adjacent leg 2 and hypotenuse 3.

(9) Find an equivalent algebraic expression for each of the following (*algebraic* meaning without any trigonometric functions).

(a) $\sin\left(\sin^{-1}(2x)\right) = 2x$

(b) $\cos\left(\sin^{-1}(2x)\right) = \sqrt{1-4x^2}$ Play the same game as in (8) (e) and (f). Draw a representative triangle and label the opposite leg $2x$ and hypotenuse 1.

(c) $\sec\left(\tan^{-1} u\right) = \sqrt{u^2+1}$ Your representative triangle can have opposite leg u and adjacent leg 1.

(10) Use the sum of angles formula for the cosine that you obtained in (5)(a) to show that if $f(x) = \cos x$, then

$$\frac{f(x+h)-f(x)}{h} = \cos x\left(\frac{\cos h - 1}{h}\right) - \sin x\left(\frac{\sin h}{h}\right)$$

This is just the trig ID with some algebra.

$$\frac{f(x+h)-f(x)}{h} = \frac{\cos(x+h)-\cos x}{h} = \frac{\cos x\cos h - \sin x\sin h - \cos x}{h}$$

$$= \frac{\cos x(\cos h - 1) - \sin x\sin h}{h} = \cos x\left(\frac{\cos h - 1}{h}\right) - \sin x\left(\frac{\sin h}{h}\right)$$

(11) Evaluate each expression exactly.

(a) $\cos\left(\ln(1)\right) = \cos(0) = 1$ (b) $\tan^{-1}\left(e^0\right) = \tan^{-1} 1 = \frac{\pi}{4}$ (c) $e^{\tan 0^\circ} = e^0 = 1$

(d) $e^{\cos 90^\circ} = e^0 = 1$ (e) $\sin^{-1}\left(\ln e\right) = \sin^{-1} 1 = \frac{\pi}{2}$ (f) $\cos^{-1}\left(\ln 1\right) = \cos^{-1} 0 = \frac{\pi}{2}$

(12) Find all solutions of the given trigonometric equation on the interval $[0, 2\pi)$.

(a) $2\sin x - \sqrt{3} = 0$ $\left\{\frac{\pi}{3}, \frac{2\pi}{3}\right\}$

(b) $\sin(2x) + \cos x = 0$ $\left\{\frac{\pi}{2}, \frac{7\pi}{6}, \frac{3\pi}{2}, \frac{11\pi}{6}\right\}$

(c) $\tan^2 x = 1$ $\left\{\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}\right\}$

(d) $2\cos^2 x - 1 = \cos x$ $\left\{0, \frac{2\pi}{3}, \frac{4\pi}{3}\right\}$

(e) $\sin x + 1 = 2\cos^2 x$ $\left\{\frac{\pi}{6}, \frac{5\pi}{6}, \frac{3\pi}{2}\right\}$ See details.

$\sin x + 1 = 2(1-\sin^2 x) \implies 2\sin^2 x + \sin x - 1 = 0$. Factor to get

$$(2\sin x - 1)(\sin x + 1) = 0 \implies \sin x = \frac{1}{2} \quad \text{or} \quad \sin x = -1.$$

The first equation has two solutions (one in each of quadrants I and II), and the second equation has one solution.

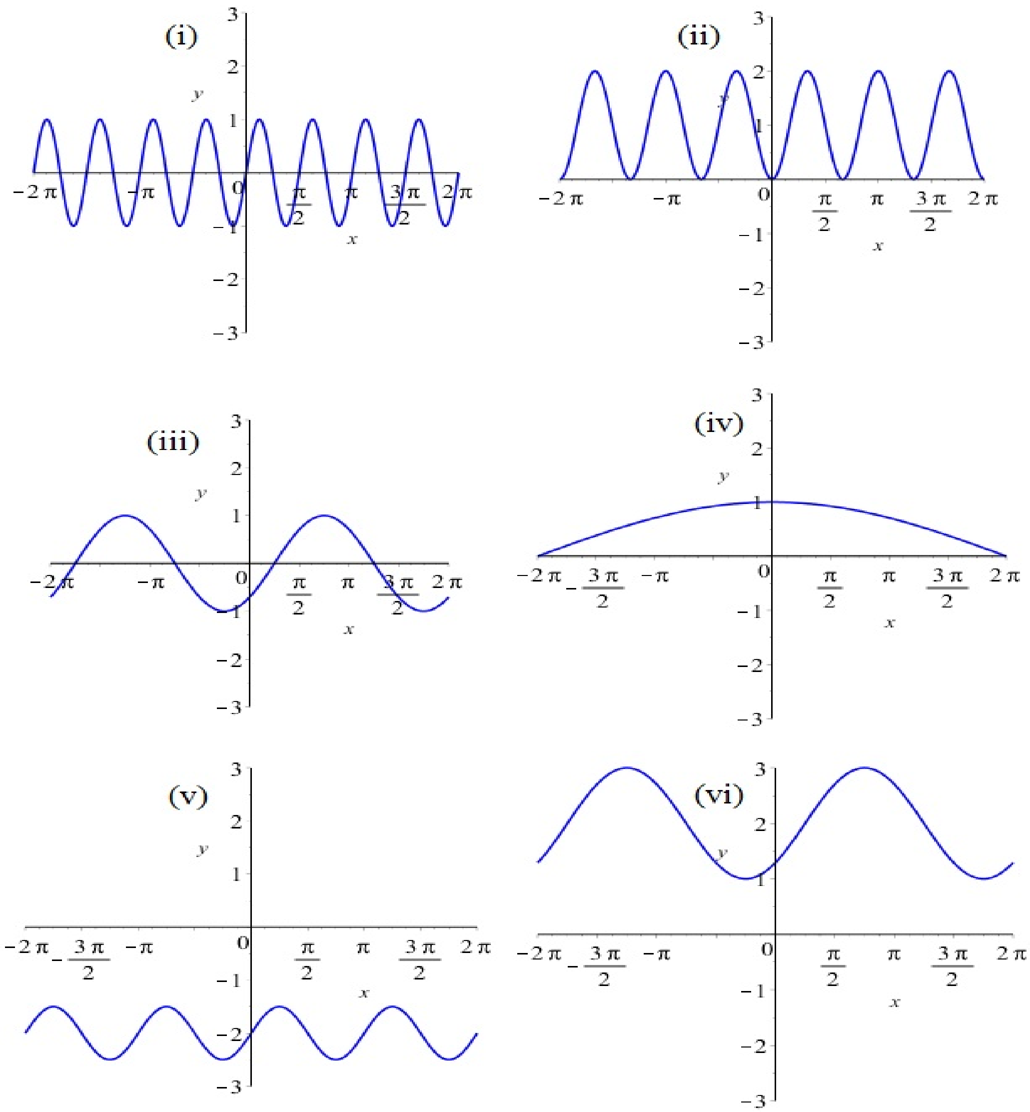
(13) Plot at least two full periods of each of $y = \sin x$, $y = \cos x$, and $y = \tan x$. Plot these and compare your graphs to the lecture slides or to an image you create in Desmos, Wolfram Alpha, or a graphing calculator.

(14) Match the following functions with the plots shown. Note that not all of the functions will be used.

(a) $f(x) = 2 - \cos\left(x + \frac{\pi}{4}\right)$ (vi) (b) $f(x) = \sin(4x)$ (i) (c) $f(x) = -2\sin(2x) + 1$ NA

(d) $f(x) = -3\cos x + 1$ NA (e) $f(x) = -\cos(3x) + 1$ (ii) (f) $f(x) = \frac{1}{2}\sin(2x) - 2$ (v)

(g) $f(x) = 2 + \cos\left(\frac{\pi x}{4} - \frac{\pi}{2}\right)$ NA (h) $f(x) = \cos\left(\frac{x}{4}\right)$ (iv) (i) $f(x) = \sin\left(x - \frac{\pi}{4}\right)$ (iii)



The following problem types would have to appear on a part of the exam in which calculator use is allowed.

(15) The hour hand on a certain clock is 4 inches long. Determine the length of the arc traversed by the tip of the hour hand between 1 pm and 6 pm (on the same day, so in five hours).

The arclength $s = r\theta$. The angle makes up $\left(\frac{5}{12}\right)^{ths}$ of the clock face (5 of the 12 hours), and each hour is $\frac{2\pi}{12} = \frac{\pi}{6}$ radians. So $\theta = \frac{5\pi}{6}$. Therefore the arclength

$$s = (4\text{in})\frac{5\pi}{6} = \frac{10\pi}{3}\text{ in} \approx 10.47\text{ in}.$$

(16) The rear wheel of a tractor has a 24 in radius. Find the angle (in radians) through which a wheel rotates in 11 sceonds if the tractor is traveling at a speed of 23 mph.

The speed is linear velocity ν. We can use the relationship $\nu = r\omega$ to find the angular velocity ω. It's useful to convert the given speed to inches per second.

$$\nu = 23\frac{\text{mi}}{\text{hr}} \cdot \frac{1\text{hr}}{3600\text{sec}} \cdot \frac{63360\text{in}}{1\text{mi}} = 404.8\frac{\text{in}}{\text{sec}}$$

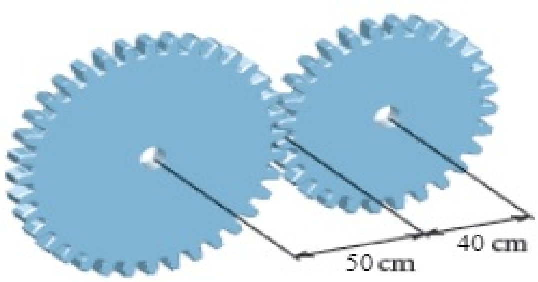
Given an radius $r = 24$ in, the angular velocity of the wheel is

$$\omega = \frac{\nu}{r} = \frac{404.8\frac{\text{in}}{\text{sec}}}{24\text{in}} = 16.8\overline{66}\frac{1}{\text{sec}}.$$

To find the angle through which the wheel rotates in a time interval of length t, we can use $\omega = \frac{\theta}{t}$. The angle in 11 seconds is

$$\theta = t\omega = 11\text{sec} \cdot 16.8\overline{66}\frac{1}{\text{sec}} \approx 185.5.$$

(17) One gear wheel turns another, the teeth being on the rims. The wheels have 40 cm and 50 cm radii, and the smaller wheel rotates at 10 rotations per minute. Find the angular speed of the larger wheel, in radians per second. (The answers is actually better if you don't bother with a calculator.)



Points on the rims of the gears share a common linear velocity ν. If we call the radius and angular velocity of the smaller gear r_1 and ω_1 and the radius and angular velocity of the larger gear r_2 and ω_2, we have

$$\nu = r_1\omega_1 = r_2\omega_2.$$

We're given the radii and $\omega_1 = 10$ rpm $= 10(2\pi)$ radians per minute. We can convert ω_1 to radians per second, and use the given radii to find ω_2. We have

$$\omega_1 = 20\pi\frac{\text{rad}}{\text{min}} \cdot \frac{1\text{min}}{60\text{sec}} = \frac{\pi}{3}\frac{\text{rad}}{\text{sec}}.$$

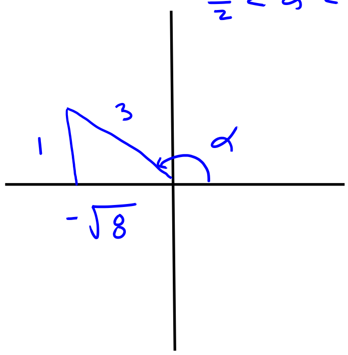
So the angular velocity of the large gear is

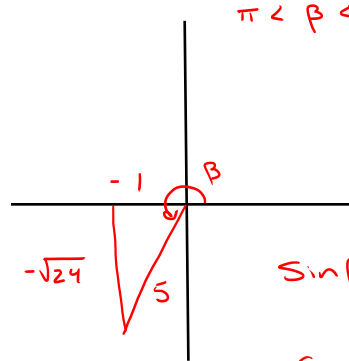
$$\omega_2 = \frac{r_1}{r_2}\omega_1 = \frac{40\text{in}}{50\text{in}}\frac{\pi}{3}\frac{\text{rad}}{\text{sec}} = \frac{4\pi}{15}\frac{\text{rad}}{\text{sec}}.$$

Problem 6 : From the given info

$\frac{\pi}{2} < \alpha < \pi$

$\pi < \beta < \frac{3\pi}{2}$





$\sin\beta = \frac{-\sqrt{24}}{5}$

$\cos\beta = \frac{-1}{5}$

$\sin\alpha = \frac{1}{3}$

$\cos\alpha = \frac{-\sqrt{8}}{3}$

a) $$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

$$= \frac{-\sqrt{8}}{3}\left(\frac{-1}{5}\right) + \frac{1}{3}\left(\frac{-\sqrt{24}}{5}\right)$$

$$= \frac{\sqrt{8} - \sqrt{24}}{15}$$

b) and c) are similarly obtained from the values given by the diagrams