Review for Exam IV

MATH 2306 (Ritter)

Sections Covered: 10, 11, 12, 13, 14, 15

This review is provided as a courtesy to give some idea of what material is covered. Nothing else is intended or implied.

(1) A certain spring is 1 ft long with no mass attached. An object weighing 10 lbs is attached to the spring. The length of the spring with the mass attached is then 18 inches.

(a) Compute the mass m in slugs and the spring constant k in lbs/ft. $m = \frac{5}{16}$ slugs $\quad k = 20$ lb/ft

(b) If the object is initially at equilibirum and given a downward velocity of 1 ft/sec, find the displacement for $t > 0$. $x(t) = -\frac{1}{8}\sin(8t)$

(c) Next assume that a driving force of $f(t) = \cos(\gamma t)$ is applied to the object. What value of γ will result in pure resonance? Resonance frequency is the natural, circular frequency $\omega = 8$ /sec.

(d) Let $f(t) = \cos(3t)$. Determine the displacement for $t > 0$ assuming the object started from rest at equilibrium . $x(t) = -\frac{16}{275}\cos(8t) + \frac{16}{275}\cos(3t)$

(2) A 64 lb object is attached to a spring whose spring constant is 26 lb/ft. A dashpot provides damping that is numerically equal to 8 times the instantaneous velocity.

(a) Determine the mass m of the object in slugs. $m = 2$ slugs

(b) Assuming there is no external applied force, set up the differential equation for the displacement and determine if the motion is overdamped, underdamped or critically damped. $2x'' + 8x' + 26x = 0$, i.e. $x'' + 4x' + 13x = 0$. The system is underdamped with characteristic roots $r = -2 \pm 3i$.

(c) If the object is initially displaced 6 inches above equilibrium and given an initial upward velocity of 2 ft/sec, determine the displacement for $t > 0$. $x(t) = \frac{1}{2}e^{-2t}\cos(3t) + e^{-2t}\sin(3t)$

(3) A 200 volt battery is applied to a series circuit with inductance 2 henries, resistance 26 ohms and capacitance $\frac{1}{80}$ farads. Find the charge on the capacitor $q(t)$ for $t > 0$ assuming the initial charge and current are zero, $q(0) = 0$, $i(0) = 0$. $q(t) = \frac{25}{6}e^{-8t} - \frac{20}{3}e^{-5t} + \frac{5}{2}$

(4) A 2 lb object is attached to a spring whose spring constant is 162 lb/ft. The system is undamped, and an external driving force of $f(t) = -4\cos(\gamma t)$ is applied. Assume that the driving force induces pure resonance. If the object starts from rest at equilibrium, determine the displacement for $t > 0$. If the spring has a maximum stretched length of 4 ft, after how many seconds will the amplitude of the oscillations exceed the maximum spring length? Resonance frequency is $\omega = \sqrt{162/2} = 9$ per second. The IVP is $x'' + 81x = -2\cos(9t)$ with $x(0) = x'(0) = 0$. The displacement is $x(t) = -\frac{1}{9}t\sin(9t)$. The amplitude $|t/9|$ will exceed 4 ft when $t > 36$ seconds.

(5) Consider the nonhomogeneous equation $x^2y'' + xy' - 4y = 20x^3$.

(a) One solution of the associated homogeneous equation is $y_1 = x^2$. Find a second linearly independent one y_2. $y_2 = \frac{1}{x^2}$

(b) Find a particular solution y_p of the nonhomogeneous equation. $y_p = 4x^3$

(c) Solve the IVP: $x^2y'' + xy' - 4y = 20x^3,\ y(1) = 3,\ y'(1) = 6$. $y = -2x^2 + \frac{1}{x^2} + 4x^3$

(6) Use the method of variation of parameters to find a particular solution for each nonhomogeneous equation.

(a) $y'' + y = \sec\theta\tan\theta \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$ $\quad y_p = \theta\cos\theta + \sin\theta\ln(\sec\theta)$

(b) $y'' + 2y + 3y' = \sin(e^x)$ $\quad y_p = -e^{-2x}\sin(e^x)$

(7) Use the definition (i.e. compute an integral) to show that $\mathscr{L}\{e^{at}\} = \frac{1}{s-a}$ for $s > a$. Use the fact that $e^{-st}e^{at} = e^{-(s-a)t}$. The convergence of the integral will require $s - a > 0$.

(8) Compute the transform or inverse transform as indicated. (Use the table of Laplace transforms along with any necessary algebra or identities.)

(a) $\mathscr{L}\{(2t-3)^2\} = \mathscr{L}\{4t^2 - 12t + 9\} = \dfrac{8}{s^3} - \dfrac{12}{s^2} + \dfrac{9}{s}$

(b) $\mathscr{L}\{\cos^2 t - \sin^2 t\} = \mathscr{L}\{\cos(2t)\} = \dfrac{s}{s^2+4}$

(c) $\mathscr{L}\{e^{3t}\sin(2t) + e^{-t}t^3\} = \dfrac{2}{(s-3)^2+4} + \dfrac{6}{(s+1)^4}$

(d) $\mathscr{L}^{-1}\left\{\dfrac{1}{s^4} - \dfrac{s}{s^2+5}\right\} = \mathscr{L}^{-1}\left\{\dfrac{1}{6}\dfrac{3!}{s^4} - \dfrac{s}{s^2+(\sqrt{5})^2}\right\} = \dfrac{1}{6}t^3 - \cos(\sqrt{5}t)$

(e) $\mathscr{L}^{-1}\left\{\dfrac{5s+3}{s^2+s}\right\} = \mathscr{L}^{-1}\left\{\dfrac{3}{s} - \dfrac{2}{s+1}\right\} = 3 + 2e^{-t}$

(f) $\mathscr{L}^{-1}\left\{\dfrac{s+8}{s^2+4s+13}\right\} = \mathscr{L}^{-1}\left\{\dfrac{s+2}{(s+2)^2+3^2} + 2\dfrac{3}{(s+2)^2+3^2}\right\} = e^{-2t}\cos(3t) + 2e^{-2t}\sin(3t)$

(9) Write each piecewise defined function in terms of appropriate unit step functions. Then find the Laplace transform of each.

(a) $f(t) = \begin{cases} t^2, & 0 \le t < 1 \\ e^t, & t \ge 1 \end{cases}$ $\quad f(t) = t^2 - t^2\mathscr{U}(t-1) + e^t\mathscr{U}(t-1)$

$$\mathscr{L}\{f(t)\} = \frac{2}{s^3} - e^{-s}\{(t+1)^2\} + e^{-s}\mathscr{L}\{e^{t+1}\} = \frac{2}{s^2} - e^{-s}\left(\frac{2}{s^3} + \frac{2}{s^2} + \frac{1}{s}\right) + e^{-s}\frac{e}{s-1}$$

(b) $f(t) = \begin{cases} 0, & 0 \le t < \frac{\pi}{4} \\ \cos(2t), & t \ge \frac{\pi}{4} \end{cases}$ $\quad f(t) = \cos(2t)\mathscr{U}\left(t - \frac{\pi}{4}\right)$

$$\mathscr{L}\{f(t)\} = e^{-\frac{\pi}{4}s}\mathscr{L}\left\{\cos\left(2\left(t + \frac{\pi}{4}\right)\right)\right\} = e^{-\frac{\pi}{4}s}\mathscr{L}\left\{\cos\left(2t + \frac{\pi}{2}\right)\right\} = e^{-\frac{\pi}{4}s}\mathscr{L}\left\{-\sin(2t)\right\} = \frac{-2e^{-\frac{\pi}{4}s}}{s^2+4}$$

(10) Find the inverse Laplace transform of each function.

(a) $\mathscr{L}^{-1}\left\{\dfrac{6e^{-2s}}{s^4}+\dfrac{e^{-s}}{s^2+1}\right\} = (t-2)^3\mathscr{U}(t-2)+\sin(t-1)\mathscr{U}(t-1)$

(b) $\mathscr{L}^{-1}\left\{\dfrac{(s-2)e^{-\pi s}}{s^2-s}\right\} = \mathscr{L}^{-1}\left\{\left(\dfrac{2}{s}-\dfrac{1}{s-1}\right)e^{-\pi s}\right\} = 2\mathscr{U}(t-\pi)+e^{t-\pi}\mathscr{U}(t-\pi)$