Review for Exam IV

MATH 2306

Sections Covered: 15, 16, 17, 18

This review is provided as a courtesy to give some idea of what material is covered. Nothing else is intended or implied.

(1) Find the Laplace transform using any method.

(a) $f(t) = e^{3t}(t-1)^2$ $\quad \mathscr{L}\{f(t)\} = \dfrac{2}{(s-3)^3} - \dfrac{2}{(s-3)^2} + \dfrac{1}{s-3}$

(b) $f(t) = t^2\mathscr{U}(t-1) - e^t\mathscr{U}(t-4)$ $\quad \mathscr{L}\{f(t)\} = \dfrac{2e^{-s}}{s^3} + \dfrac{2e^{-s}}{s^2} + \dfrac{e^{-s}}{s} - \dfrac{e^4e^{-4s}}{s-1}$

(c) $f(t) = \begin{cases} 2t, & 0 \le t < 3 \\ 1, & 3 \le t \end{cases} = 2t - 2t\mathscr{U}(t-3) + \mathscr{U}(t-3), \quad \mathscr{L}\{f(t)\} = \dfrac{2}{s^2} - \dfrac{2e^{-3s}}{s^2} - \dfrac{5e^{-3s}}{s}$

(2) Find the inverse Laplace transform using any method.

(a) $F(s) = \dfrac{s}{s^2-4s+10}$ $\quad \mathscr{L}^{-1}\{F(s)\} = e^{2t}\cos(\sqrt{6}t) + \dfrac{2}{\sqrt{6}}e^{2t}\sin(\sqrt{6}t)$

(b) $F(s) = \dfrac{2s+5}{(s-3)^2}$ $\quad \mathscr{L}^{-1}\{F(s)\} = 2e^{3t} + 11te^{3t}$

(c) $F(s) = \dfrac{3e^{-2s}}{s(s+1)^2}$ $\quad \mathscr{L}^{-1}\{F(s)\} = 3\mathscr{U}(t-2) - 3e^{-(t-2)}\mathscr{U}(t-2) - 3(t-2)e^{-(t-2)}\mathscr{U}(t-2)$

(3) Solve the IVP using the Laplace transform.

(a) $y'' - 2y' + 5y = 0, \quad y(0) = 2, \quad y'(0) = 4$ $\quad y = 2e^t\cos(2t) + e^t\sin(2t)$

(b) $y'' + 3y' - 4y = 80t, \quad y(0) = 1, \quad y'(0) = -4$ $\quad y = -15 - 20t + 16e^t$

(c) $y'' + 4y' + 4y = 42t^5e^{-2t} \quad y(0) = 1, \quad y'(0) = 0$ $\quad y = t^7e^{-2t} + 2te^{-2t} + e^{-2t}$

(4) Solve the IVP using the Laplace transform.

$$y'' + y = \mathscr{U}\left(t - \frac{\pi}{4}\right), \quad y(0) = 0, \quad y'(0) = 2$$

$$y = \mathscr{U}\left(t - \frac{\pi}{4}\right) - \cos\left(t - \frac{\pi}{4}\right)\mathscr{U}\left(t - \frac{\pi}{4}\right) + 2\sin t$$

(5) An LRC series circuit has inductance 1 h, resistance 2 ohms and capacitance 0.1 f. The initial charge on the capacitor and current in the circuit are $q(0) = i(0) = 0$. At time $t = 0$, a unit pulse voltage is applied to the circuit so that the charge satisfies

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = \delta(t).$$

The *function* $\delta(t)$ satisfies $\mathscr{L}\{\delta(t)\} = 1$. Find the charge on the capacitor q for $t > 0$ using the method of Laplace transforms. $q(t) = \frac{1}{3}e^{-t}\sin(3t)$ [1]

(6) Suppose f is a function such that $f(0) = 1$ and $\mathscr{L}\{f'(t)\} = \frac{\ln s}{s}$. Determine $\mathscr{L}\{f(t)\}$. (In the words of Dennis Zill, "Don't think deep thoughts.") $\mathscr{L}\{f(t)\} = \frac{\ln s}{s^2} + \frac{1}{s}$

(7) Find the Fourier series of the given function

(a) $f(x) = 1, \quad -\pi < x < \pi$ $\quad f(x) = 1$ (Yeah, that's all!)

(b) $f(x) = \begin{cases} 0, & -2 < x < 0 \\ 2x, & 0 \le x < 2 \end{cases}$ $\quad f(x) = 1 + \sum_{n=1}^{\infty}\left[\frac{4((-1)^n - 1)}{n^2\pi^2}\cos\left(\frac{n\pi x}{2}\right) + \frac{4(-1)^{n+1}}{n\pi}\sin\left(\frac{n\pi x}{2}\right)\right]$

(c) $f(x) = \begin{cases} -x - 1, & -1 < x < 0 \\ 1 - x, & 0 \le x < 1 \end{cases}$ $\quad f(x) = \sum_{n=1}^{\infty}\frac{2}{n\pi}\sin(n\pi x)$

(8) Consider the function in (7)(b) above. What does the series you found converge to at $x = 0$, $x = 1$, $x = 2$, $x = 4$ and $x = 6$? f is continuous at 0 and 1 so the series converges to $f(0) = 0$

[1] Technically speaking, this should be multiplied by the unit step centered at zero, $\mathscr{U}(t)$. This solution is valid for $t > 0$, but the solution is not differentiable in the traditional sense at zero.

at $x = 0$ and to $f(1) = 2$ at $x = 1$. The periodic extention with period 4 would have a jump discontinuity at $x = 2$ with left limit $f(2^-) = 4$ and right limit $f(-2^+) = 0$. So the series converges in the mean to $\dfrac{4+0}{2} = 2$ at $x = 2$. The periodic extension would converge to $f(0) = 0$ when $x = 4$ and again converge in the mean to $\dfrac{4+0}{2} = 2$ at $x = 6$.

(8) Consider the function $f(x) = \begin{cases} 2x, & 0 \le x < \frac{1}{2} \\ 1, & \frac{1}{2} \le x < 1 \end{cases}$. Give a plot of the half range sine series and a plot of the half range cosine series of f over the interval $[-3, 3]$. Find each of these series.

$$f(x) = \frac{3}{4} + \sum_{n=1}^{\infty} \left[\frac{4}{n^2\pi^2} \left(\cos\left(\frac{n\pi}{2}\right) - 1 \right) \cos(n\pi x) \right]$$

$$f(x) = \sum_{n=1}^{\infty} \left(\frac{4}{n^2\pi^2} \sin\left(\frac{n\pi}{2}\right) - \frac{2(-1)^n}{n\pi} \right) \sin(n\pi x)$$

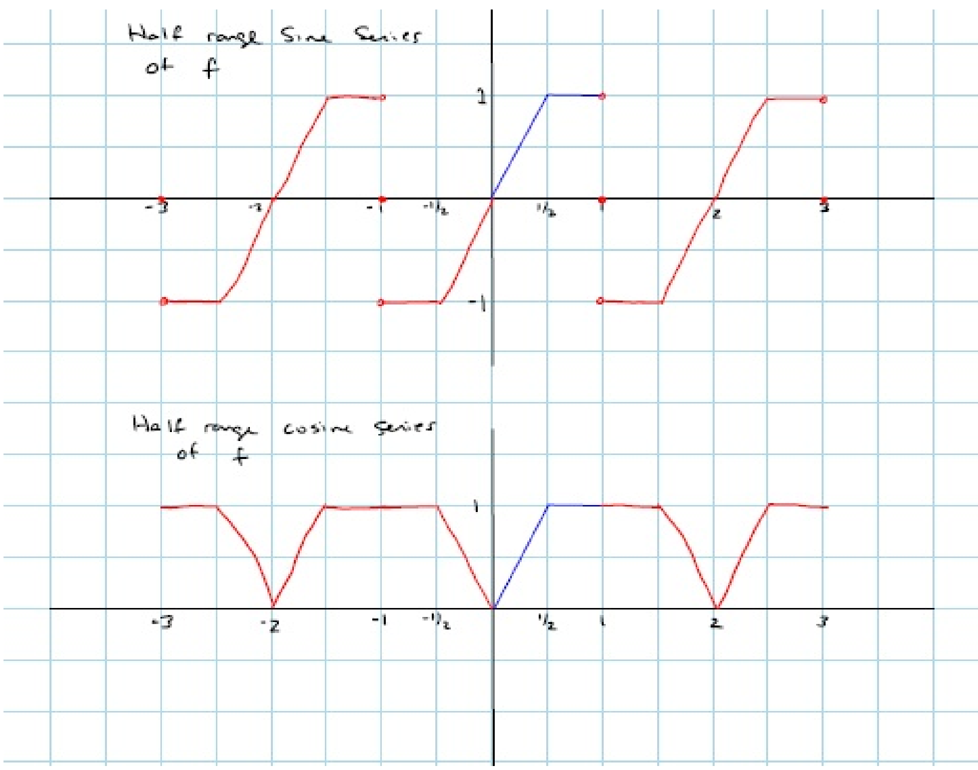

Figure 1: In each figure, the part of the curve shown in blue is the plot of $f(x)$. Note the open and closed points on the top plot showing *convergence in the mean*.