(1) Consider the linear system of equations
\[
\begin{align*}
  x_1 + 3x_2 &= -1 \\
-2x_1 + bx_2 &= 7
\end{align*}
\]
for some constant $b$.

Determine all values of $b$ for which the system is inconsistent.

\[
\begin{bmatrix}
1 & 3 & -1 \\
-2 & b & 7
\end{bmatrix}
\xrightarrow{R_1+R_2 \rightarrow R_2}
\begin{bmatrix}
1 & 3 & -1 \\
0 & b+6 & 5
\end{bmatrix}
\]

The 2nd equation is $(b+6)x_2 = 5$.

If $b = -6$, this is
\[
0 = 5
\]
otherwise it's solvable with $x_2 = \frac{5}{b+6}$.

(2) Now consider the linear system of equations
\[
\begin{align*}
  x_1 + 3x_2 &= c \\
-2x_1 + bx_2 &= 7
\end{align*}
\]
for some constants $b$ and $c$.

Determine all conditions on the values of $b$ and $c$ such that the system has

(i) exactly one solution
\[
\begin{bmatrix}
1 & 3 & c \\
-2 & b & 7
\end{bmatrix}
\xrightarrow{R_1+R_2 \rightarrow R_2}
\begin{bmatrix}
1 & 3 & c \\
0 & b+6 & 2c+7
\end{bmatrix}
\]

Now the 2nd equation is $(b+6)x_2 = 2c+7$.

(i) If $b \neq -6$ there is exactly one solution with
\[
x_2 = \frac{2c+7}{b+6}.
\]
This is always well defined.

(ii) If $b = -6$, the equation is $0 = 2c+7$.
This is false if $2c+7$ is not zero.
But, if $2c+7 = 0$, the equation is $0 = 0$ which is always true.

So if $b = -6$ and $c = -\frac{7}{2}$, $x_2$ is free and there are infinitely many solutions.