Math 3260 Practice - Section 4.1

Names: ________________________________

(1) Determine whether each set is a (real) vector space. If not, which axiom(s) fail(s) to hold?

(a) The set of all $3 \times 2$ matrices whose second row is all zeros with standard matrix addition and scalar multiplication.

This is a vector space. $M_{3\times 2}$, the set of all real $3\times 2$ matrices, is a vector space. The current space is a subspace. In particular, $\mathbf{0}$ is in the set since its second row is zeros. If $A = [a_{ij}]$ and $B = [b_{ij}]$ are in this set, then $a_{ij} = 0$, $b_{ij} = 0$ so $a_{ij} + b_{ij} = 0$ and $ca_{ij} = 0$ for $j=1,2$.

Hence it's closed under vector addition and scalar multiplication.

(b) The set of all polynomials of degree $\geq 2$ together with $0$ with regular addition and scalar multiplication of functions.

This is not a vector space. It is not closed under vector addition. For example, $p_1(t) = t^2$ and $p_2(t) = 1 - t^2$ are in the set. But

$$(p_1 + p_2)(t) = p_1(t) + p_2(t) = t^2 + (1 - t^2) = 1$$

is not (its degree is < 2).
(2) Consider the set of two formal symbols \( \{x, y\} \) where \( x \neq y \). Define vector addition by

\[
 x + x = x, \quad y + y = x, \quad \text{and} \quad x + y = y + x = y.
\]

Define scalar multiplication by

\[
 cx = x \quad \text{and} \quad cy = y, \quad \text{for all scalars } c.
\]

Check to see whether this set with these operations satisfies each of the vector space axioms.

1) This holds by the definition of the addition

2) Yes addition commutes as given

3) \((x + x) + x = x + x = x + (x + x)\)

\((x + x) + y = x + y = x + (x + y)\)

\((x + y) + z = x + y = x + (y + z)\)

\((y + z) + x = y + x = y + (z + x)\)

We have associativity! (This is all possible combos.)

4) \(x + x = x\) and \(x + y = y\) so \(x\) serves as \(0\)

5) \(x + x = x\) and \(y + y = y\) so \(x\) is \(-x\) and \(y\) is \(-y\).

6) We get this for free by the way scalar mult. is defined.

7) \(c(x + x) = c\cdot x = x\) \(; \) \(c + c\cdot x = x + x = x\)

\(c(x + y) = c\cdot y = y\) \(; \) \(c\cdot x + c\cdot y = x + y = y\)

\(c(y + z) = c\cdot z = z\) \(; \) \(c\cdot y + c\cdot z = y + z = z\)

So property 7 holds (again this is all possible combos).

8) \((c + d)x = x\) \(; \) \(c\cdot x + d\cdot x = x + x = x\)

\((c + d)y = y\) \(; \) \(c\cdot y + d\cdot y = y + y = y\) These don't match!

Property 8 fails. Scalar multiplication does not distribute over addition of scalars.
9) \( c(\lambda x) = \lambda c x = \lambda \bar{x}, \quad d(c \lambda x) = c \lambda \bar{x} = \lambda \bar{x} \quad (cd) \lambda x = \lambda \bar{x} \)
\( c(d \lambda y) = \lambda c \bar{y} = \lambda \bar{y}, \quad d(c \lambda y) = c \lambda \bar{y} = \lambda \bar{y} \quad (cd) \lambda y = \lambda \bar{y} \)
Property 9 holds.

10) \( 1 \cdot x = x, \quad 1 \cdot \bar{y} = \bar{y} \)
Property 10 holds.

This is not a vector space, but only Property 9 fails.