September 10 MATH 1113 sec. 51 Fall 2018

Section 3.2 & 3.3: Quadratic Functions and Quadratic Equations

Factorable and Irreducible If a quadratic polynomial $f(x) = ax^2 + bx + c$ has real zeros x_0 and x_1 (not necessarily distinct), then it can be factored and written as

$$f(x) = a(x - x_0)(x - x_1).$$

If f has no real zeros, which can be determined for example by looking at the discriminant, then f is said to be irreducible.

This is exhaustive! That is, every quadratic is either factorable as a product of linear factors or it is irreducible.

Section 4.3: Polynomial Division, Remainders & Factors

Suppose we wish to plot a polynomial such as $f(x) = -x^3 + 6x^2 - 9x$. One step in the process is finding intercepts.

It would be helpful to know that

$$f(x) = -x(x^2 - 6x + 9) = -x(x-3)^2.$$

In particular, from the factored form it is easy to see that because $x - 3$ is a factor of f, $f(3) = 0$.

$f(3) = -3(3-3)^2 = -3 \cdot 0^2 = 0$

Polynomial Division

We can use long division to determine if one polynomial (usually first degree) is a factor of another. Given a polynomial P of degree n and a polynomial d of degree k with $k < n$, we can write

$$P(x) = d(x)Q(x) + R(x)$$

where Q and R are polynomials[1]. The degree of Q is $n - k$ and the degree of R is less than k.

[1] P dividend, d divisor, Q quotient, R remainder

Example

Divide $f(x) = 3x^3 + 11x^2 - 2x + 8$ by (a) $x - 1$, and by (b) $x + 4$.

The division is just like division of integers but using powers of x instead of powers of 10.

ⓐ

$$
\begin{array}{r|l}
 & 3x^2 + 14x + 12 \\
\hline
x - 1 & 3x^3 + 11x^2 - 2x + 8 \\
 & -(3x^3 - 3x^2) \\
\hline
 & 14x^2 - 2x \\
 & -(14x^2 - 14x) \\
\hline
 & 12x + 8 \\
 & -(12x - 12) \\
\hline
\end{array}
$$

$3x^3 = x(3x^2)$

$14x^2 = x(14x)$

$12x = x(12)$

20

Degree is zero
This is the remainder

This tells us that

$$3x^3+11x^2-2x+8=(x-1)(3x^2+14x+12)+20$$

P d Q R

This can also be written as

$$\frac{3x^3+11x^2-2x+8}{x-1}=3x^2+14x+12+\frac{20}{x-1}$$

(b)

$$\begin{array}{r|l} & 3x^2 - x + 2 \\ \hline x+4 & 3x^3 + 11x^2 - 2x + 8 \\ & -(3x^3 + 12x^2) \\ \hline & \quad -x^2 - 2x \\ & \quad -(-x^2 - 4x) \\ \hline & \qquad 2x + 8 \\ & \qquad -(2x + 8) \\ \hline & \qquad\quad 0 \end{array}$$

$\leftarrow$ Remainder

$$3x^3 + 11x^2 - 2x + 8 = (x+4)(3x^2 - x + 2)$$

Question

Find the quotient $Q(x)$ and the remainder $R(x)$ from the division

$$(3x^4 - x^3 - 2x^2 + 2x - 1) \div (x - 2)$$

(a) $Q(x) = 3x^3 - 7x^2 + 12x - 22$, and $R(x) = 43$

(b) $Q(x) = 3x^3 - 7x^2 + 12x$, and $R(x) = -22x - 1$

(c) $Q(x) = 3x^3 + 5x^2 + 8x$, and $R(x) = 18x - 1$

(d) $Q(x) = 3x^3 + 5x^2 + 8x + 18$, and $R(x) = 35$

(e) I know how to do this, but my answer is not here.

The Remainder Theorem

Recall that we found that $x + 4$ is a factor of $f(x) = 3x^3 + 11x^2 - 2x + 8$ (remainder 0), and $x - 1$ was not (remainder 20) . In fact, we can note that $f(-4) = 0$ and $f(1) = 20$. This illustrates the following theorem.

Theorem:
If the polynomial

$f(c) = (c-c)Q(c) + R$
$= 0 + R = R$

$$f(x) = (x - c)Q(x) + R,$$

then $f(c) = R$. That is, $f(c)$ is the remainder when f is divided by the factor $x - c$.

Corollary: The Factor Theorem

Theorem:
For polynomial f, $f(c) = 0$ if and only if $x - c$ is a factor of f.

Question

Suppose f is a polynomial and $f(7) = 0$.Which of the following must be true?

(a) f has x-intercept $(7, 0)$.

(b) The remainder when f is divided by $x - 7$ is zero.

(c) $x - 7$ is a factor of f.

(d) All of the above are true.

(e) None of the above is true.

Section 4.1: Polynomials of Degree $n \geq 2$

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0,$$

End Behavior: What happens as $x \to \infty$ and $x \to -\infty$

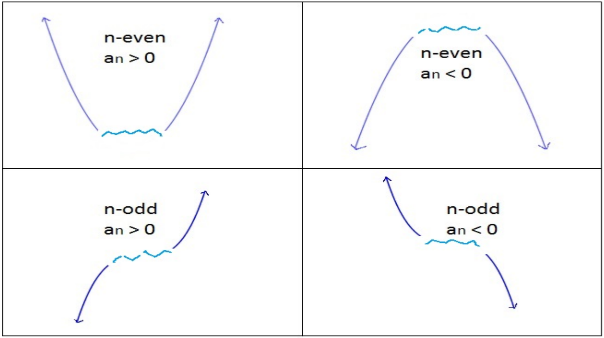
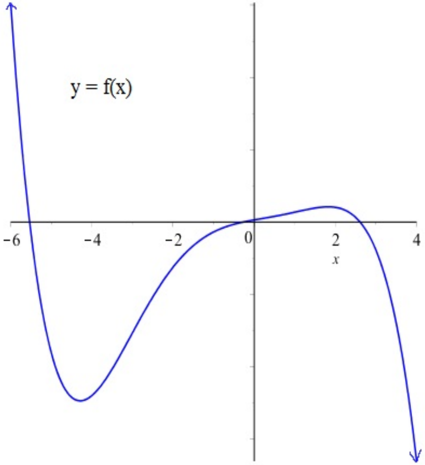


Figure: The ends of the graph go up or down. The behavior is determined by the degree n and the sign of the leading coefficient a_n.

Question

Given the graph, which of the following can be true of the polynomial f?



(a) The degree of f is 4

(b) $f(x) = 3x^3 - 14x^2 + 2x + 1$

(c) The leading coefficient of f is greater than 27

(d) The leading term of f is $-\frac{1}{3}x^5$

Roots, Zeros, and x-intercepts

Given $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$,

- a number x_0 such that $f(x_0) = 0$ is called a **zero** of the function f,
- the number x_0 is a **root** of the polynomial equation $f(x) = 0$, and
- the point $(x_0, 0)$ is an x**-intercept** on the graph of the function f.

Finding the x-intercepts of a line or quadratic is a snap! For higher degree polynomials, it may be quite difficult.

We may be able to use some theorems on polynomial zeros. It may be that technological assistance is require.

Example

Find all of the x-intercepts on the graph of $f(x) = x^3 - 4x^2 + 4x$.

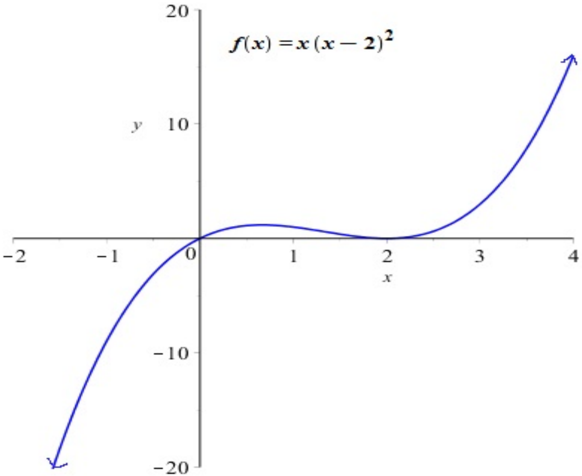
Solve $f(x) = 0$ (by factoring)

$$0 = x^3 - 4x^2 + 4x = x(x^2 - 4x + 4) = x(x-2)^2$$

$f(x) = 0$ if $x = 0$ or $(x-2)^2 = 0$

$x = 0$ or $x = 2$

There are two x-intercepts $(0,0)$ and $(2,0)$.



$f(x) = x(x-2)^2$
20
10
y
−10
−20
−2
−1
0
1
2
3
4
x