September 10 MATH 1113 sec. 52 Fall 2018

Section 3.2 & 3.3: Quadratic Functions and Quadratic Equations

Discriminant Given $ax^2 + bx + c$, the **discriminant** of the quadratic is the number

$$b^2 - 4ac.$$

Theorem: The quadratic equation $ax^2 + bx + c = 0$ has

(a) no real solutions if $b^2 - 4ac < 0$

(b) one real solution if $b^2 - 4ac = 0$, and

(c) two distinct real solutions if $b^2 - 4ac > 0$.

Definition: A quadratic polynomial is called **irreducible** if it's discriminant is negative.

Question

The discriminant of $3x^2 - 12x + 7$ is

(a) 60

$b^2 - 4ac = (-12)^2 - 4 \cdot 3 \cdot 7$

$= 60$

(b) -228

(c) -60

(d) 228

Question

If $f(x) = 0 \Rightarrow (x-h)^2 = -k$

Suppose $f(x) = (x-h)^2 + k$ and $k > 0$ (i.e. k is positive). Which of the following must be true about the x-intercepts of f?

$f(x) = x^2 - 2hx + h^2 + k$ $\quad b = -2h, \; a = 1, \; c = h^2 + k$

(a) The point (h, k) is the only x-intercept.

$b^2 - 4ac = (-2h)^2 - 4(1)(h^2+k)$

(b) f has two different x-intercepts.

$= 4h^2 - 4h^2 - 4k = -4k$

(c) f doesn't have any x-intercepts because the discriminant is $-4k$ which is negative.

(d) Nothing can be said about x-intercepts without knowing the sign of h.

Factorable and Irreducible

If a quadratic polynomial $f(x) = ax^2 + bx + c$ has real zeros x_0 and x_1 (not necessarily distinct), then it can be factored and written as

$$f(x) = a(x - x_0)(x - x_1).$$

If f has no real zeros, which can be determined for example by looking at the discriminant, then f is said to be irreducible.

This is exhaustive! That is, every quadratic is either factorable as a product of linear factors or it is irreducible.

Section 4.3: Polynomial Division, Remainders & Factors

Suppose we wish to plot a polynomial such as $f(x) = -x^3 + 6x^2 - 9x$. One step in the process is finding intercepts.

It would be helpful to know that

$$f(x) = -x(x^2 - 6x + 9) = -x(x - 3)^2.$$

In particular, from the factored form it is easy to see that because $x - 3$ is a factor of f, $f(3) = 0$.

$f(3) = -3(3-3)^2 = -3 \cdot 0^2 = 0$

Polynomial Division

We can use long division to determine if one polynomial (usually first degree) is a factor of another. Given a polynomial P of degree n and a polynomial d of degree k with $k < n$, we can write

$$P(x) = d(x)Q(x) + R(x)$$

where Q and R are polynomials[1]. The degree of Q is $n - k$ and the degree of R is less than k.

[1] P dividend, d divisor, Q quotient, R remainder

Example

Divide $f(x) = 3x^3 + 11x^2 - 2x + 8$ by (a) $x - 1$, and by (b) $x + 4$.

Long division is 'just' like long division with integers except that the "places" are powers of x as opposed to powers of 10.

(a)

$$
\begin{array}{r|l}
 & 3x^2 + 14x + 12 \\
\hline
x-1 & 3x^3 + 11x^2 - 2x + 8 \\
 & -(3x^3 - 3x^2) \\
\hline
 & \quad 14x^2 - 2x \\
 & \quad -(14x^2 - 14x) \\
\hline
\end{array}
$$

$3x^3 = x(3x^2)$

$14x^2 = x(14x)$

$$\begin{array}{r} 12x + 8 \\ -\ (12x - 12) \\ \hline 20 \end{array}$$

$12x = x(12)$

This is the remainder the degree is smaller than 1

This is interpreted as

$$\underbrace{3x^3+11x^2-2x+8}_{P} = \underbrace{(x-1)}_{d}\underbrace{(3x^2+14x+12)}_{Q} + \underbrace{20}_{R}$$

Also, for $x \neq 1$

$$\frac{3x^3+11x^2-2x+8}{x-1} = 3x^2+14x+12 + \frac{20}{x-1}$$

(b)

$$
\begin{array}{r|l}
 & 3x^2 - x + 2 \\
\hline
x+4 & 3x^3 + 11x^2 - 2x + 8 \\
 & -(3x^3 + 12x^2) \\
\hline
 & -x^2 - 2x \\
 & -(-x^2 - 4x) \\
\hline
 & 2x + 8 \\
 & -(2x + 8) \\
\hline
 & 0
\end{array}
$$

$$\underbrace{3x^3 + 11x^2 - 2x + 8}_{P} = \underbrace{(x+4)}_{d} \underbrace{(3x^2 - x + 2)}_{Q}$$

R is zero

Question

Find the quotient $Q(x)$ and the remainder $R(x)$ from the division

$$(3x^4 - x^3 - 2x^2 + 2x - 1) \div (x - 2)$$

(a) $Q(x) = 3x^3 - 7x^2 + 12x - 22$, and $R(x) = 43$

(b) $Q(x) = 3x^3 - 7x^2 + 12x$, and $R(x) = -22x - 1$

(c) $Q(x) = 3x^3 + 5x^2 + 8x$, and $R(x) = 18x - 1$

(d) $Q(x) = 3x^3 + 5x^2 + 8x + 18$, and $R(x) = 35$

(e) I know how to do this, but my answer is not here.

The Remainder Theorem

Recall that we found that $x+4$ is a factor of $f(x) = 3x^3 + 11x^2 - 2x + 8$ (remainder 0), and $x-1$ was not (remainder 20) . In fact, we can note that $f(-4) = 0$ and $f(1) = 20$. This illustrates the following theorem.

Theorem:
If the polynomial

$$f(x) = (x-c)Q(x) + R,$$

then $f(c) = R$. That is, $f(c)$ is the remainder when f is divided by the factor $x-c$.

$f(c) = (c-c)Q(c) + R = 0 + R = R$

Corollary: The Factor Theorem

Theorem:
For polynomial f, $f(c) = 0$ if and only if $x - c$ is a factor of f.

Question

Suppose f is a polynomial and $f(7) = 0$.Which of the following must be true?

(a) f has x-intercept $(7, 0)$.

(b) The remainder when f is divided by $x - 7$ is zero.

(c) $x - 7$ is a factor of f.

(d) All of the above are true.

(e) None of the above is true.