Sept 10 Math 2306 sec. 53 Fall 2018

Section 5: First Order Equations Models and Applications

RC Series Circuit: The charge $q(t)$ on the capacitor of an RC-series circuit with resistance R, capacitance C, and implied electromotive force E is governed by

$$R\frac{dq}{dt} + \frac{1}{C}q = E(t).$$

LR Series Circuit: The current $i = \frac{dq}{dt}$ in an LR-series circuit with resistance R, inductance L, and implied electromotive force E is governed by

$$L\frac{di}{dt} + Ri = E(t).$$

Measurable Quantities & Units

Resistance R in ohms (Ω),	Implied voltage E in volts (V),
Inductance L in henries (h),	Charge q in coulombs (C),
Capacitance C in farads (f),	Current i in amperes (A)

Current is the rate of change of charge with respect to time: $i = \dfrac{dq}{dt}$.

Example

A 200 volt battery is applied to an RC series circuit with resistance 1000Ω and capacitance 5×10^{-6} f. Find the charge $q(t)$ on the capacitor if $i(0) = 0.4A$. Determine the charge as $t \to \infty$.

$$R\frac{dq}{dt} + \frac{1}{C}q = E \qquad R = 1000\Omega, \quad C = 5 \cdot 10^{-6} f$$

$$i(0) = 0.4 = q'(0)$$

$$1000\frac{dq}{dt} + \frac{1}{5 \cdot 10^{-6}}q = 200 \qquad E(t) = 200$$

Get it in standard form:

$$\frac{dq}{dt} + \frac{10^6}{5(1000)}q = \frac{200}{1000}$$

$$\frac{10^6}{5(10^3)} = \frac{10^3}{5} = 200$$

$$\frac{dq}{dt} + 200q = \frac{1}{5}, \quad q'(0) = \frac{2}{5}$$

Get Integrating factor:

$$P(t) = 200, \quad \mu = e^{\int P(t)\,dt} = e^{\int 200\,dt} = e^{200t}$$

$$\frac{d}{dt}\left(e^{200t} q\right) = \frac{1}{5} e^{200t}$$

$$\int \frac{d}{dt}\left(e^{200t} q\right) dt = \int \frac{1}{5} e^{200t}\,dt$$

$$e^{200t} q = \frac{1}{5} \cdot \frac{1}{200} e^{200t} + K$$

$$q = \frac{1}{1000} + k e^{-200t}$$

Apply $q'(0) = \frac{2}{5}$ $\qquad q'(t) = 0 + k e^{-200t}(-200)$

$$@_{t=0} \quad \frac{2}{5} = -200k\, e^{0} = -200k$$

$$k = \frac{-2}{5(200)} = \frac{-1}{500}$$

The charge q on the capacitor is given by

$$q(t) = \frac{1}{1000} - \frac{1}{500} e^{-200t}$$

Letting $t \to \infty$, the long term charge

$$\lim_{t\to\infty} q(t) = \lim_{t\to\infty}\left(\frac{1}{1000} - \frac{1}{500}e^{-200t}\right) = \frac{1}{1000}$$

A Classic Mixing Problem

A tank originally contains 500 gallons of pure water. Brine containing 2 pounds of salt per gallon is pumped in at a rate of 5gal/min. The well mixed solution is pumped out at the same rate. Find the amount of salt $A(t)$ in pounds at the time t. Find the concentration of the mixture in the tank at $t = 5$ minutes.

A Classic Mixing Problem

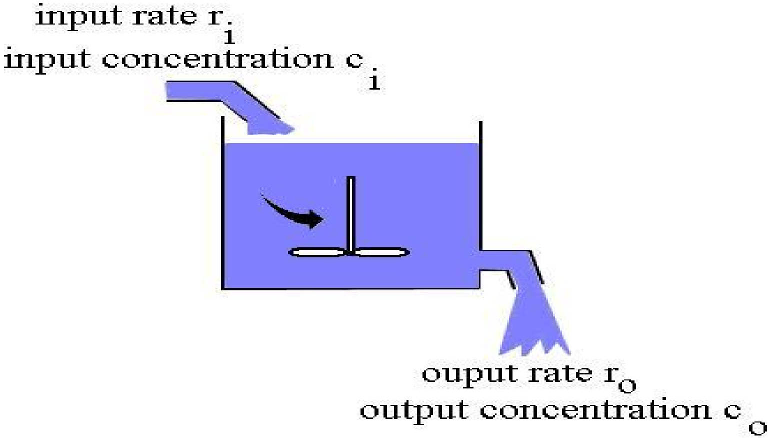

Figure: Spatially uniform composite fluids (e.g. salt & water, gas & ethanol) being mixed. Concentrations of substance change in time.

Building an Equation

The rate of change of the amount of salt

$$\frac{dA}{dt} = \left(\begin{array}{c}\textit{input rate}\\ \textit{of salt}\end{array}\right) - \left(\begin{array}{c}\textit{output rate}\\ \textit{of salt}\end{array}\right)$$

The input rate of salt is

$$\text{fluid rate in} \cdot \text{concentration of inflow} \quad = \quad r_i(c_i).$$

The output rate of salt is

$$\text{fluid rate out} \cdot \text{concentration of outflow} \quad = \quad r_o(c_o).$$

Building an Equation

= concentration in the tank

The concentration of the outflowing fluid is

$$\frac{\text{total salt}}{\text{total volume}} = \frac{A(t)}{V(t)} = \frac{A(t)}{V(0)+(r_i-r_o)t}.$$

$$\frac{dA}{dt} = r_i \cdot c_i - r_o \frac{A}{V}.$$

This equation is first order linear.

Rearrange

$$\frac{dA}{dt} + \frac{r_o}{V} A = r_i c_i$$

1st order linear

Solve the Mixing Problem

A tank originally contains 500 gallons of pure water. Brine containing 2 pounds of salt per gallon is pumped in at a rate of 5gal/min. The well mixed solution is pumped out at the same rate. Find the amount of salt $A(t)$ in pounds at the time t. Find the concentration of the mixture in the tank at $t = 5$ minutes.

$$\frac{dA}{dt} + \frac{r_o}{V} A = r_i c_i \qquad A(0) = A_0$$

$r_i = 5$ gal/min $\qquad c_i = 2 \frac{\text{lb}}{\text{gal}}$

$r_o = 5$ gal/min $\qquad V(0) = 500$ gal $\qquad$ so

$$V = 500\,\text{gal} + \left(5\,\frac{\text{gal}}{\text{min}} - 5\,\frac{\text{gal}}{\text{min}}\right) t\ \text{min}$$

$= 500\ \text{gal}$

pure water $\Rightarrow$ $A(0) = 0$

$$\frac{dA}{dt} + \frac{5\ \frac{\text{gal}}{\text{min}}}{500\ \text{gal}} A = 5\ \frac{\text{gal}}{\text{min}} \cdot 2\ \frac{\text{lb}}{\text{gal}}$$

$$\frac{dA}{dt} + \frac{1}{100} A = 10 \quad , \quad A(0) = 0$$

Integrating factor $P(t) = \frac{1}{100}$, $\mu = e^{\int P(t)\,dt}$

$$= e^{\int \frac{1}{100}\,dt} = e^{\frac{1}{100}t}$$

$$\frac{d}{dt}\left(e^{\frac{1}{100}t} A\right) = 10\, e^{\frac{1}{100}t}$$

$$\int \frac{d}{dt}\left(e^{\frac{1}{100}t} A\right) dt = \int 10\, e^{\frac{1}{100}t}\, dt$$

$$e^{\frac{1}{100}t} A = 10(100)\, e^{\frac{1}{100}t} + k$$

$$A = 1000 + k\, e^{\frac{-1}{100}t}$$

using $A(0) = 0$ @ $t = 0$

$$0 = 1000 + k e^{0}$$

$$k = -1000$$

The amount of salt

$$\boxed{A(t) = 1000 - 1000 e^{\frac{-1}{100} t}}$$

After 5 minutes, the concentration in the tank is

$$\frac{A(5)}{V(5)} = \frac{1000 - 1000 e^{\frac{-1}{100}(5)}}{500} \approx 0.01 \ \frac{\text{lb}}{\text{gal}}$$