September 12 MATH 1113 sec. 51 Fall 2018

Section 4.1: Polynomials of Degree $n \geq 2$

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0,$$

End Behavior: What happens as $x \to \infty$ and $x \to -\infty$

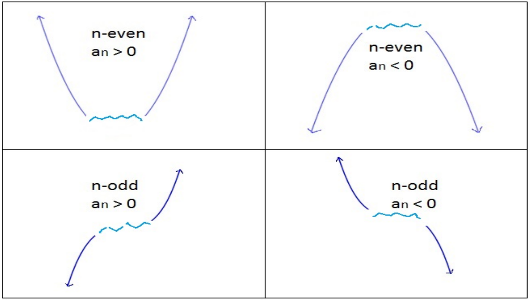


Figure: The ends of the graph go up or down. The behavior is determined by the degree n and the sign of the leading coefficient a_n.

Roots, Zeros, and x-intercepts

Given $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$,

- a number x_0 such that $f(x_0) = 0$ is called a **zero** of the function f,
- the number x_0 is a **root** of the polynomial equation $f(x) = 0$, and
- the point $(x_0, 0)$ is an x**-intercept** on the graph of the function f.

Finding the x-intercepts of a line or quadratic is a snap! For higher degree polynomials, it may be quite difficult.

We may be able to use some theorems on polynomial zeros. It may be that technological assistance is require.

Example

Find all of the x-intercepts on the graph of $f(x) = x^3 - 4x^2 + 4x$.

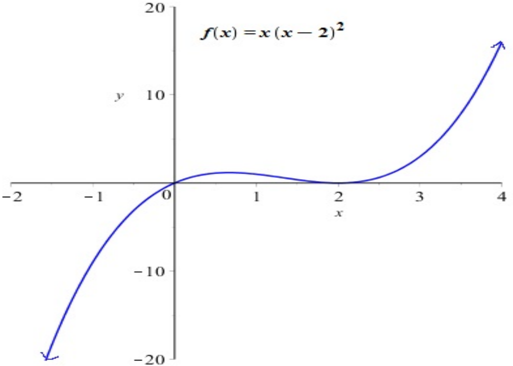
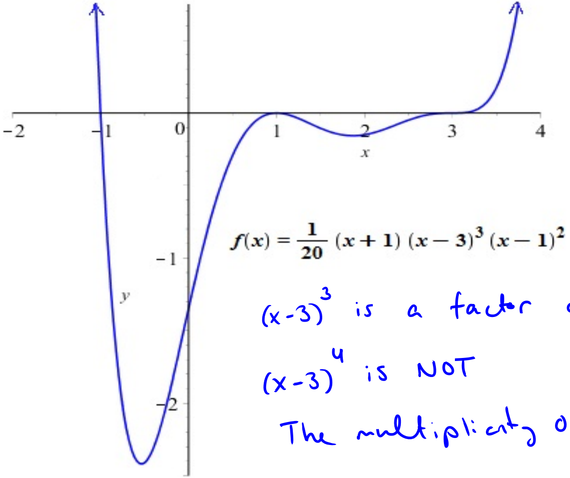


Figure: We did this last time and found that $f(x) = x(x-2)^2$ so that there are two x-intercepts $(0,0)$ and $(2,0)$.

Multiplicity of Zeros



$(x-3)^3$ is a factor of f

$(x-3)^4$ is NOT

The multiplicity of 3 is 3.

Figure: The function $f(x) = \frac{1}{20}x^6 - 10x^5 + 35x^4 - 44x^3 - 9x^2 + 54x - 27$. Note that it has three x-intercepts. It **crosses** the axis at -1 and 3. It is **tangent** to the axis at 1.

Multiplicity of Zeros

Definition: Suppose that $(x-c)^k$ is a factor of a polynomial $P(x)$ and $(x-c)^{k+1}$ is not a factor of P. Then c is called a **zero of multiplicity** k of the function P. We may also call it a **root of multiplicity** k of the polynomial.

Why do we care about multiplicity?
Well, there are various reasons you'll encounter, but we'll start with a graph related one.

Multiplicity of Zeros

Theorem: Suppose f is a polynomial function and c is a zero of multiplicity k. Then $(c, 0)$ is an x-intercept to the graph of f, and

- if k is odd, the graph **crosses** the axis at $(c, 0)$,
- if k is even, the graph is **tangent** but does not cross the axis at $(c, 0)$

Good News! It may seem tough to remember this result, but all it really requires is that we keep a couple of basic pictures in mind. And they're from simple functions we know well.

k-odd (1, 3, etc.) looks like a line or cubic
k-even (2, 4, etc.) looks like a parabola

Multiplicity of Zeros $(x - c)^k$

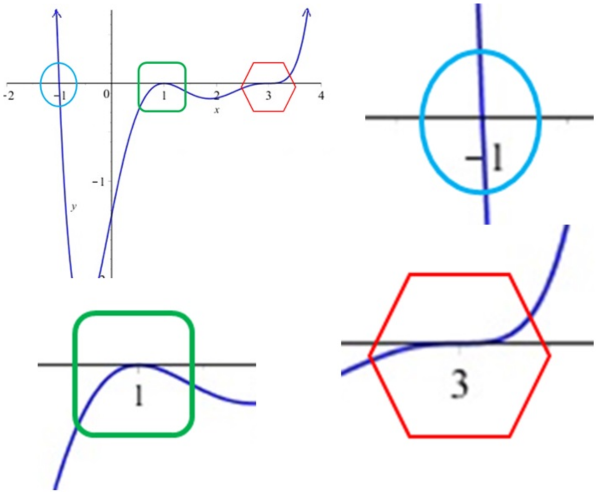
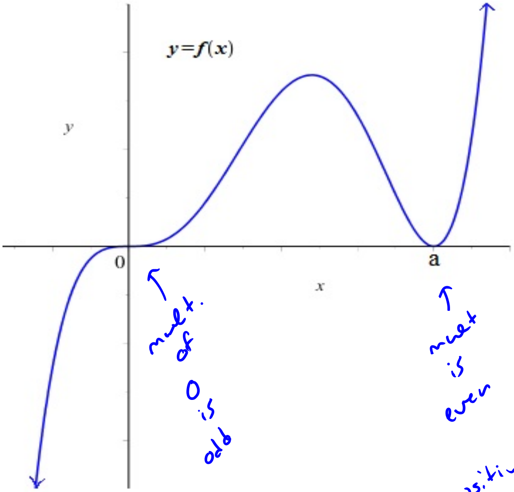


Figure: Close up, an odd root likes like a line or cubic, and an even one looks like a parabola.

Question

Which of the following functions could be shown in the plot?



(a) $f(x) = -x^3(x-a)^4$

(b) $f(x) = x^3(x-a)^4$

(c) $f(x) = x^2(x-a)^3$

(d) $f(x) = -x^2(x-a)^3$

positive leading term

The Fundamental Theorem of Algebra

Theorem: Every polynomial of degree n with real coefficients has exactly n complex zeros.

Some comments:

- The zeros are not necessarily distinct. For example $f(x) = (x-2)^4$ four zeros, but they are all the one number 2. (This is called *accounting for multiplicity*.
- The zeros are not necessarily real numbers. For example $f(x) = x^2 + 1$ has two zeros i and $-i$. These are both complex numbers.
- Every polynomial of odd degree must have at least one real number zero. **WHY?** (hint: think about the end behavior)
- The number of distinct real zeros CANNOT exceed the degree.

Minute Exercise

Plot the functions $p(x) = 1$ and $q(x) = 0$. Write a few sentences addressing the questions:

- How may zeros—x-intercepts—does p have? How many does q have?
- How does this related to the fact that the degree of p is zero, but the degree of q is not defined?