September 12 MATH 1113 sec. 52 Fall 2018

Section 4.1: Polynomials of Degree $n \geq 2$

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0,$$

End Behavior: What happens as $x \to \infty$ and $x \to -\infty$

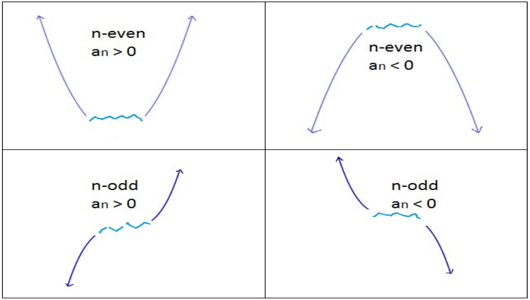
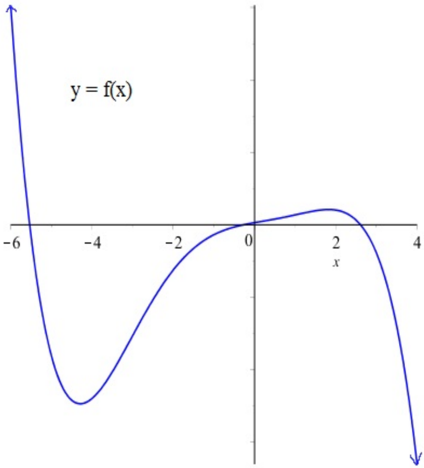


Figure: The ends of the graph go up or down. The behavior is determined by the degree n and the sign of the leading coefficient a_n.

Question

Given the graph, which of the following can be true of the polynomial f?



(a) The degree of f is 4

(b) $f(x) = 3x^3 - 14x^2 + 2x + 1$

(c) The leading coefficient of f is greater than 27

(d) The leading term of f is $-\frac{1}{3}x^5$

Roots, Zeros, and x-intercepts

Given $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$,

- a number x_0 such that $f(x_0) = 0$ is called a **zero** of the function f,
- the number x_0 is a **root** of the polynomial equation $f(x) = 0$, and
- the point $(x_0, 0)$ is an x**-intercept** on the graph of the function f.

Finding the x-intercepts of a line or quadratic is a snap! For higher degree polynomials, it may be quite difficult.

We may be able to use some theorems on polynomial zeros. It may be that technological assistance is require.

Example

Find all of the x-intercepts on the graph of $f(x) = x^3 - 4x^2 + 4x$.

$$f(x) = x^3 - 4x^2 + 4x = x(x^2 - 4x + 4) = x(x-2)^2$$

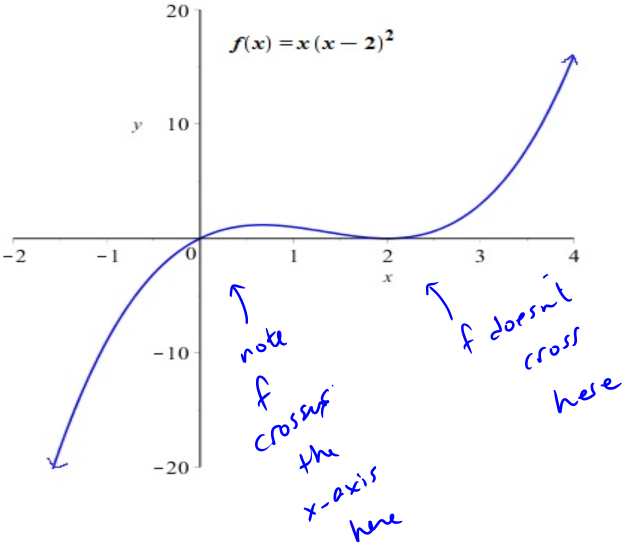
We need zeros of f. Solve $f(x) = 0$.

$0 = x(x-2)^2$ by the zero product property

$x = 0$ or $x - 2 = 0$

$x = 2$

There are two x-intercepts $(0, 0)$ and $(2, 0)$.


$f(x) = x(x-2)^2$
20
10
-10
-20
y
-2
-1
0
1
2
3
4
x
note f crosses the x-axis here
f doesn't cross here

Multiplicity of Zeros

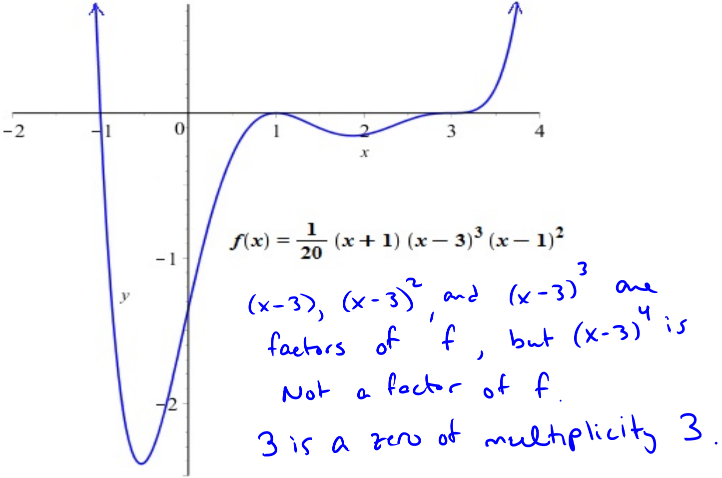


Figure: The function $f(x) = \frac{1}{20}x^6 - 10x^5 + 35x^4 - 44x^3 - 9x^2 + 54x - 27$. Note that it has three x-intercepts. It **crosses** the axis at -1 and 3. It is **tangent** to the axis at 1.

Multiplicity of Zeros

Definition: Suppose that $(x - c)^k$ is a factor of a polynomial $P(x)$ and $(x - c)^{k+1}$ is not a factor of P. Then c is called a **zero of multiplicity** k of the function P. We may also call it a **root of multiplicity** k of the polynomial.

Why do we care about multiplicity?

Well, there are various reasons you'll encounter, but we'll start with a graph related one.

Multiplicity of Zeros

Theorem: Suppose f is a polynomial function and c is a zero of multiplicity k. Then $(c, 0)$ is an x-intercept to the graph of f, and

- if k is odd, the graph **crosses** the axis at $(c, 0)$,
- if k is even, the graph is **tangent** but does not cross the axis at $(c, 0)$

Good News! It may seem tough to remember this result, but all it really requires is that we keep a couple of basic pictures in mind. And they're from simple functions we know well.

k-odd (1, 3, etc.) looks like a line or cubic
k-even (2, 4, etc.) looks like a parabola

Multiplicity of Zeros $(x - c)^k$

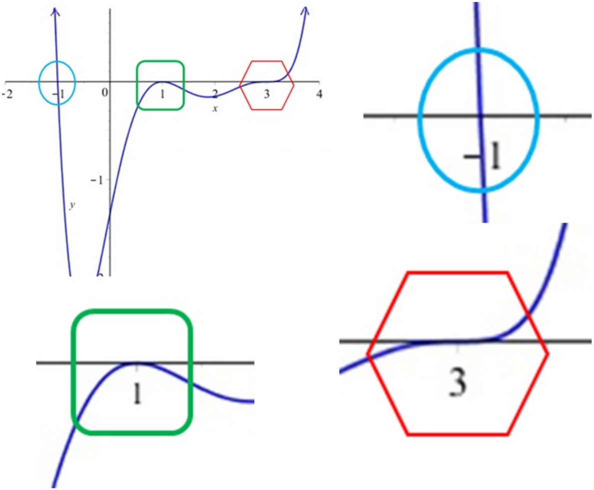
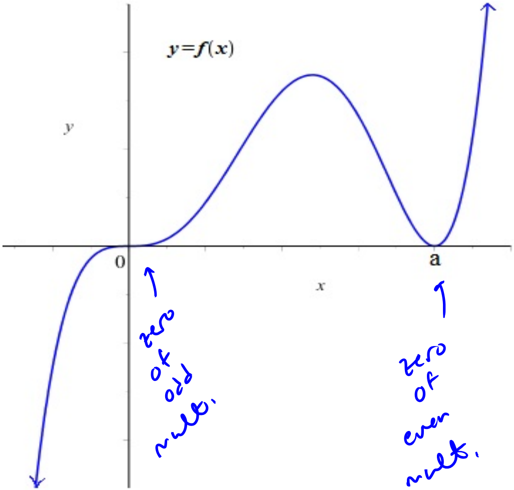


Figure: Close up, an odd root likes like a line or cubic, and an even one looks like a parabola.

Question

Which of the following functions could be shown in the plot?



(a) $f(x) = -x^3(x-a)^4$

(b) $f(x) = x^3(x-a)^4$

(c) $f(x) = x^2(x-a)^3$

(d) $f(x) = -x^2(x-a)^3$

End behavior ⇒ leading coef. is +