Sept 12 Math 2306 sec. 53 Fall 2018

Section 5: First Order Equations Models and Applications

A Classic Mixing Problem If we are tracking the amount A of some substance (e.g. salt) disolved in some fluid in which we know the flow rates at which fluid is entering (r_i) and leaving (r_o) a receptacle, the initial volume $V(0)$ of fluid, and the substance concentration of the inflow c_i, then for a **well mixed** solution

$$\frac{dA}{dt} = r_i \cdot c_i - r_o \frac{A}{V}.$$

Here $V(t) = V(0) + (r_i - r_o)t$. This equation is first order linear

$$\frac{dA}{dt} + \frac{r_o}{V} A = r_i c_i$$

There is nothing precluding one of the coefficeints such as r_o, r_i or c_i from being a nonconstant function of time.

Mixing Problem w/ Non-constant Volume

A tank originally contains 500 gallons of pure water. Brine containing 2 pounds of salt per gallon is pumped in at a rate of 5 gal/min. The well mixed solution is pumped out at the rate of 10 gal/min. Set up the initial value problem governing the amount (pounds) of salt $A(t)$ at time t in minutes. Determine the time interval for which the differential equation is valid.

$$\frac{dA}{dt} = r_i C_i - r_o C_o \quad \text{where} \quad C_o = \frac{A}{V}$$

Here $r_i = 5$ gal/min, $C_i = 2 \frac{\text{lb}}{\text{gal}}$

$r_o = 10$ gal/min

$$C_o = \frac{A}{V(0) + (r_i - r_o)t} = \frac{A}{500 + (5-10)t} = \frac{A}{500 - 5t}$$

Note C_0 is only valid until $V=0$

$$500-5t=0 \Rightarrow t=100$$

$$\frac{dA}{dt}=10-10\,\frac{A}{500-5t}=10-\frac{10}{500-5t}A$$

$$\frac{dA}{dt}+\frac{2}{100-t}A=10 \quad \text{with } A(0)=0$$

valid for $0\leq t<100$

Mixing Problem w/ Non-constant Volume

A tank originally contains 500 gallons of pure water. Brine containing 2 pounds of salt per gallon is pumped in at a rate of 5 gal/min. The well mixed solution is pumped out at the rate of 10 gal/min. Set up the initial value problem governing the amount of salt A. Determine the time interval for which the differential equation is valid.

Solving the problem is left as an exercise. The correct solution will be

$$A(t) = 10(100-t) - \frac{(100-t)^2}{10} \quad \text{valid for} \quad 0 \le t < 100$$

A Nonlinear Modeling Problem

A population $P(t)$ of tilapia changes at a rate jointly proportional to the current population and the difference between the constant carrying capacity[1] M of the environment and the current population. Determine the differential equation satsified by P.

Current population P, difference between M and P

P $\qquad$ $M-P$

$$\frac{dP}{dt} \propto P(M-P)$$

$$\Rightarrow \quad \frac{dP}{dt} = kP(M-P) \quad \text{for some constant } k$$

[1] The carrying capacity is the maximum number of individuals that the environment can support due to limitation of space and resources.

Logistic Differential Equation

The equation

$$\frac{dP}{dt} = kP(M-P), \quad k, M > 0$$

is called a **logistic growth equation.**

Solve this equation[2] and show that for any $P(0) \neq 0$, $P \to M$ as $t \to \infty$.

The ODE is separable

$$\frac{1}{P(M-P)}\frac{dP}{dt} = k \quad \Rightarrow \quad \frac{1}{P(M-P)}\,dP = k\,dt$$

$$\int \frac{1}{P(M-P)}\,dP = \int k\,dt$$

[2]The partial fraction decomposition

$$\frac{1}{P(M-P)} = \frac{1}{M}\left(\frac{1}{P} + \frac{1}{M-P}\right)$$

is useful.

Using the partial fraction decomp

$$\int \frac{1}{M}\left(\frac{1}{P} + \frac{1}{M-P}\right) dP = \int k\, dt$$

$$\int \left(\frac{1}{P} + \frac{1}{M-P}\right) dP = \int kM\, dt$$

$$\ln P - \ln|M-P| = kMt + C$$

Using log properties

$$\ln\left|\frac{P}{M-P}\right| = kMt + C$$

$$\left|\frac{P}{M-P}\right| = e^{kMt+C} = e^{C}\, e^{kMt}$$

Let $A = e^c$ or $-e^c$

$$\frac{P}{M-P} = A e^{kMt}$$

Let $P(0) = P_0$, then applying this condition

$$\frac{P_0}{M-P_0} = A e^0 = A \Rightarrow A = \frac{P_0}{M-P_0}$$

Save this

$$\frac{P}{M-P} = Ae^{kMt} \Rightarrow P = A e^{kMt}(M-P)$$

$$P = AMe^{kMt} - APe^{kMt}$$

$$P + APe^{kMt} = AMe^{kMt}$$

$$(1+Ae^{kMt})P = AMe^{kMt}$$

$$P = \frac{AMe^{kMt}}{1+Ae^{kMt}}$$

using $A = \frac{P_0}{M-P_0}$

$$P = \frac{\frac{P_0}{M-P_0} M e^{kMt}}{1 + \frac{P_0}{M-P_0} e^{kMt}} \cdot \left(\frac{M-P_0}{M-P_0}\right)$$

$$P(t) = \frac{P_0 M e^{kMt}}{M - P_0 + P_0 e^{kMt}}$$

This is the solution to the logistic equation subject to $P(0) = P_0$

The long time population

$$\lim_{t\to\infty} P(t) = \lim_{t\to\infty} \frac{P_0 M e^{kMt}}{M - P_0 + P_0 e^{kMt}} = \text{“} \frac{\infty}{\infty} \text{”}$$

by l'Hospitel's rule

$$= \lim_{t\to\infty} \frac{\cancel{P_0} M \cancel{kM} \cancel{e^{kMt}}}{\cancel{P_0} \cancel{kM} \cancel{e^{kMt}}}$$

$$= \lim_{t \to \infty} M = M$$

The limit is M as expected.

Section 6: Linear Equations Theory and Terminology

Recall that an n^{th} order linear IVP consists of an equation

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

to solve subject to conditions

$$y(x_0) = y_0, \quad y'(x_0) = y_1, \quad \ldots, \quad y^{(n-1)}(x_0) = y_{n-1}.$$

The problem is called **homogeneous** if $g(x) \equiv 0$. Otherwise it is called **nonhomogeneous**.

Theorem: Existence & Uniqueness

Theorem: If $a_0, \ldots, a_n$ and g are continuous on an interval I, $a_n(x) \neq 0$ for each x in I, and x_0 is any point in I, then for any choice of constants $y_0, \ldots, y_{n-1}$, the IVP has a unique solution $y(x)$ on I.

Put differently, we're guaranteed to have a solution exist, and it is the only one there is!

Homogeneous Equations

We'll consider the equation

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = 0$$

and assume that each a_i is continuous and a_n is never zero on the interval of interest.

Theorem: If $y_1, y_2, \ldots, y_k$ are all solutions of this homogeneous equation on an interval I, then the *linear combination*

$$y(x) = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_k y_k(x)$$

is also a solution on I for any choice of constants $c_1, \ldots, c_k$.

This is called the **principle of superposition.**

Corollaries

(i) If y_1 solves the homogeneous equation, the any constant multiple $y = cy_1$ is also a solution.

(ii) The solution $y = 0$ (called the trivial solution) is always a solution to a homogeneous equation.

Big Questions:

- Does an equation have any **nontrivial** solution(s), and
- since y_1 and cy_1 aren't truly *different* solutions, what criteria will be used to call solutions distinct?

Linear Dependence

Definition: A set of functions $f_1(x), f_2(x), \dots, f_n(x)$ are said to be **linearly dependent** on an interval I if there exists a set of constants $c_1, c_2, \dots, c_n$ with at least one of them being nonzero such that

$$c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x) = 0 \quad \text{for all} \quad x \text{ in } I.$$

A set of functions that is not linearly dependent on I is said to be **linearly independent** on I.