September 14 MATH 1113 sec. 52 Fall 2018

Section 4.1: Polynomials of Degree $n \geq 2$

Theorem: (The Fundamental Theorem of Algebra) Every polynomial of degree n with real coefficients has exactly n complex zeros.

Some comments:

- The zeros are not necessarily distinct. For example $f(x) = (x-2)^4$ four zeros, but they are all the one number 2. (This is called *accounting for multiplicity*.
- The zeros are not necessarily real numbers. For example $f(x) = x^2 + 1$ has two zeros i and $-i$. These are both complex numbers.
- Every polynomial of odd degree must have at least one real number zero. **WHY?** (hint: think about the end behavior)
- The number of distinct real zeros CANNOT exceed the degree.

Minute Exercise

Plot the functions $p(x) = 1$ and $q(x) = 0$. Write a few sentences addressing the questions:

- How may zeros—x-intercepts—does p have? How many does q have?
- How does this related to the fact that the degree of p is zero, but the degree of q is not defined?

Section 4.2: Graphing a Polynomial

Turning Points

Definition: A **turning point** on the graph of a function f is a point at which f changes from increasing to decreasing or from decreasing to increasing. Note that a turning point is a relative maximum or a relative minimum.

Theorem: If $P(x)$ is a polynomial of degree n, then the graph of P has at most $n - 1$ turning points.

There are Caclulus tools that can help you to find the location of turning points.

Recap on Polynomial Features

We want to graph and recognize graphs of polynomials. Here's a laundry list of things we know about a polynomial P of degree n.

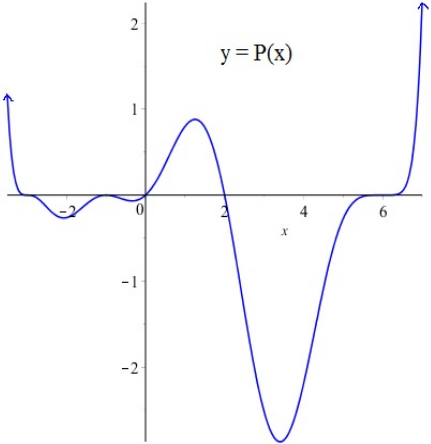
- The domain is all reals, and the graphs are continuous and smooth.
- The degree and leading coefficient determine how the graph behaves at the far left and right.
- There are at most n x-intercepts, and exactly n complex zeros accounting for multiplicity.
- There are at most $n-1$ turning points.
- The graph crosses (odd) or is tangent (even) to the x-axis based on the multiplicity of the zero.
- Are we missing anything?

Question

Consider the plot of the polynomial P. Note the end behavior of P and that P has five x-intercepts.

True or False: The degree of P is an even number that is greater than or equal to 6.

True



(a) True, but I'm not confident.

(b) True, and I am confident.

(c) False, but I'm not confident.

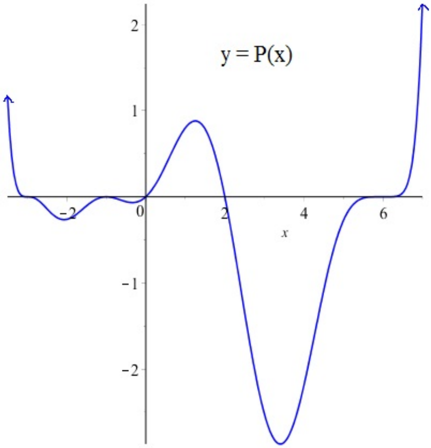
(d) False, and I am confident.

Question

Consider the plot of the polynomial P. Note the end behavior of P and that P has five x-intercepts.

True or False: The leading coefficient of P must be negative.

False



(a) True, but I'm not confident.

(b) True, and I am confident.

(c) False, but I'm not confident.

(d) False, and I am confident.

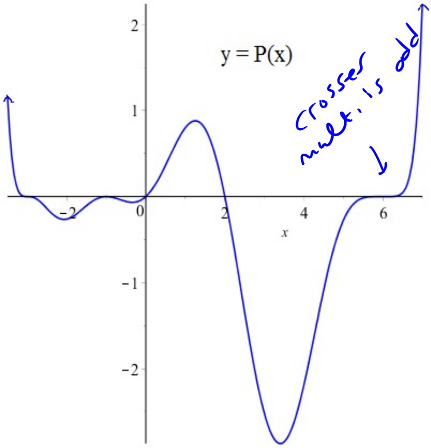
Question

Consider the plot of the polynomial P. The graph of P contains the point $(6, 0)$.

$P(x) = \cdots (x-6)^k \cdots$

True or False: 6 is a zero of P, and it's multiplicity is even.

False



crosses mult. is odd

(a) True, but I'm not confident.

(b) True, and I am confident.

(c) False, but I'm not confident.

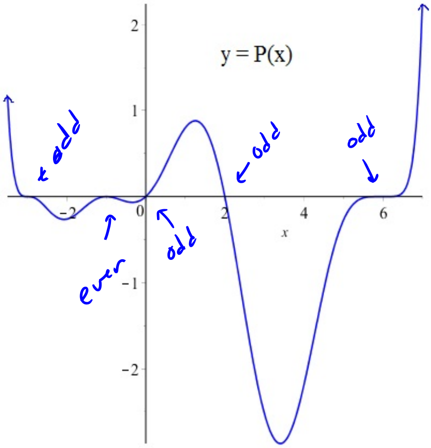
(d) False, and I am confident.

Question

Consider the plot of the polynomial P. The zeros of P are -3, -1, 0, 2 and 6.

True or False: Only one of the zeros of P has even multiplicity.

True @ -1



(a) True, but I'm not confident.

(b) True, and I am confident.

(c) False, but I'm not confident.

(d) False, and I am confident.

Graphing a Polynomial

Sketch a plot of the graph of $f(x) = -x^3 + 6x^2 - 9x$.

We'll do the following preliminary work, then use the information to sketch.

1. Determine the end behavior.
2. Find the y-intercept $(0, f(0))$
3. Find the x-intercepts. Make note of multiplicity.
4. Divide the x-axis into intervals according to the zeros, and check the sign of f in each interval.
5. Plot the intercepts and points used in the previous step.
6. Cross reference the facts (number of roots, zero multiplicities, maximum number of turning points), and sketch.

$f(x) = -x^3 + 6x^2 - 9x$

There is a table a couple of slides away to keep track of the relevant details.

end behavior

The leading term is $-x^3$ if this is $a_n x^n$

then $a_n = -1$ and $n = 3$

negative odd

Looks like ↑ stuff ↓

y-intercept

$f(0) = -0^3 + 6 \cdot 0^2 - 9 \cdot 0 = 0$

f has y-intercept $(0,0)$

$$f(x) = -x^3 + 6x^2 - 9x$$

x-intercepts

$$f(x) = -x^3 + 6x^2 - 9x = -x\left(x^2 - 6x + 9\right)$$

$$= -x(x-3)^2$$

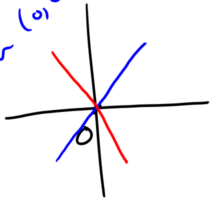
mult. 1 mult. 2

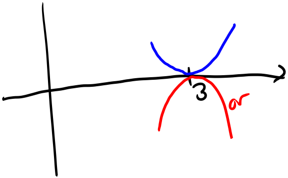
$$f(x) = 0 \Rightarrow -x(x-3)^2 = 0 \Rightarrow x = 0 \quad \text{or} \quad x - 3 = 0$$

$$x = 3$$

There are two x-intercepts $(0,0)$ and $(3,0)$.

graph near $(0,0)$





graph near $(3,0)$

$$f(x) = -x^3 + 6x^2 - 9x$$

We'll divide the domain using the x-intercepts. We have

$$-\infty < x < 0, \quad 0 < x < 3 \quad \text{and}$$

$$3 < x < \infty.$$

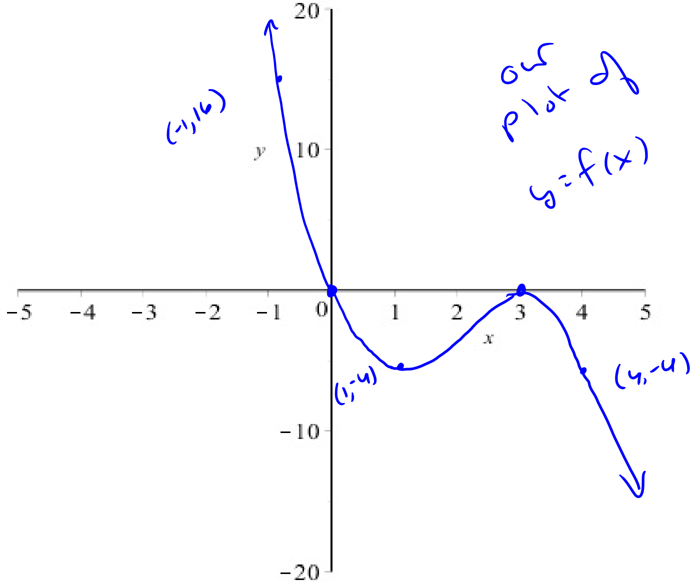
Testing points:

$$f(-1) = -(-1)^2 + 6(-1)^2 - 9(-1) = 1 + 6 + 9 = 16$$

$$f(1) = -(1)^3 + 6(1)^2 - 9(1) = -1 + 6 - 9 = -4$$

$$f(4) = -4^3 + 6(4)^2 - 9(4) = -64 + 96 - 36 = -4$$

Interval	$(-\infty, 0)$	$(0, 3)$	$(3, \infty)$			
test pt c	-1	1	4			
$f(c)$	$f(-1) = 16$	$f(1) = -4$	$f(4) = -4$			
sign	pos.	neg.	neg			


20
10
y
-10
-20
-5
-4
-3
-2
-1
0
1
2
3
4
5
x
(-1,16)
(1,-4)
(4,-4)
our plot of
y = f(x)