Sept. 14 Math 1190 sec. 51 Fall 2016

Section 2.1: Rates of Change and the Derivative

Let $y = f(x)$. For $x \neq c$ we'll call $\frac{f(x)-f(c)}{x-c}$ the average rate of change of f on the interval from x to c.

We'll call

$$\lim_{x\to c} \frac{f(x)-f(c)}{x-c} \quad \text{the rate of change of } f \text{ at } c$$

if this limit exists.

Definition: Let $y = f(x)$ at let c be in the domain of f. The **derivative** of f at c is denoted $f'(c)$ and is defined as

$$f'(c) = \lim_{x\to c} \frac{f(x)-f(c)}{x-c}$$

provided the limit exists.

The Derivative

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

In addition to *the derivative of* f *at* c, the notation $f'(c)$ is read as

- f prime of c, or
- f prime at c.

At this point, we have several interpretations of this same **number** $f'(c)$.

- as a velocity if f is the position of a moving object,
- as a rate of change of the function f when $x = c$,
- as the slope of the line tangent to the graph of f at $(c, f(c))$.

Example

Find $g'(2)$ if $g(x) = x^4$.

By defintion

$$g'(2) = \lim_{x \to 2} \frac{g(x) - g(2)}{x - 2}$$

$$g(2) = 2^4 = 16$$

$$= \lim_{x \to 2} \frac{x^4 - 16}{x - 2}$$

$$= \lim_{x \to 2} \frac{(x^2 - 4)(x^2 + 4)}{x - 2}$$

$$= \lim_{x \to 2} \frac{\cancel{(x-2)}(x+2)(x^2+4)}{\cancel{x-2}}$$

$$= \lim_{x \to 2} (x+2)(x^2+4)$$

$$= (2+2)(2^2+4) = 4(8) = 32$$

$$g'(2) = 32$$

Section 2.2: The Derivative as a Function

If $f(x)$ is a function, then the set of numbers $f'(c)$ for various values of c can define a new function. To proceed, we consider an alternative formulation for $f'(c)$.

If it exists, then $f'(c) = \lim_{x \to c} \frac{f(x)-f(c)}{x-c}$. Let $h = x - c$. Then $h \to 0$ if $x \to c$, and $x = c + h$. Hence we can write $f'(c)$ as

$$f'(c) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}.$$

The Derivative Function

Let f be a function. Define the new function f' by

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

called the **derivative** of f. The domain of this new function is the set

$$\{x | x \text{ is in the domain of } f\text{, and } f'(x) \text{ exists}\}.$$

f' is read as "f prime."

Example

Let $f(x) = \sqrt{x-1}$. Identify the domain of f. Find f' and identify its domain.

For $f(x)$, we require $x - 1 \geq 0 \Rightarrow x \geq 1$.

The domain of f is $[1, \infty)$.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\sqrt{x+h-1} - \sqrt{x-1}}{h}$$

use conjugate

$$= \lim_{h \to 0} \left(\frac{\sqrt{x+h-1} - \sqrt{x-1}}{h} \right) \left(\frac{\sqrt{x+h-1} + \sqrt{x-1}}{\sqrt{x+h-1} + \sqrt{x-1}} \right)$$

$$= \lim_{h \to 0} \frac{\cancel{x} + h - \cancel{1} - (\cancel{x} - \cancel{1})}{h\left(\sqrt{x+h-1} + \sqrt{x-1}\right)}$$

$$= \lim_{h \to 0} \frac{\cancel{h}}{\cancel{h}\left(\sqrt{x+h-1} + \sqrt{x-1}\right)}$$

$$= \lim_{h \to 0} \frac{1}{\sqrt{x+h-1} + \sqrt{x-1}}$$

$$= \frac{1}{\sqrt{x+0-1} + \sqrt{x-1}} = \frac{1}{2\sqrt{x-1}}$$

$$f'(x) = \frac{1}{2\sqrt{x-1}}$$

For the domain, we require $x - 1 > 0$

$\Rightarrow \quad x > 1.$

The domain of f' is $(1, \infty)$.

Note: The domain of f was $[1, \infty)$. The domain of f' is a little bit different.

* $[1, \infty)$ indicates $1 \leq x < \infty$

where as

$(1, \infty)$ indicates $1 < x < \infty$

Example Continued...

Use the results to find the equation of the line tangent to the graph of $f(x) = \sqrt{x-1}$ at the point $(2,1)$.

Recall that the slope of the tangent line @ $(2,1)$ is $f'(2)$.

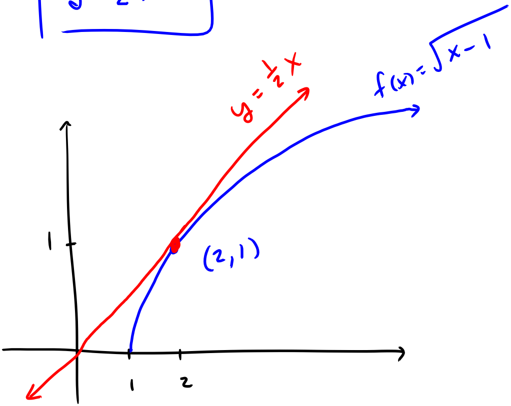
$f'(x) = \frac{1}{2\sqrt{x-1}}$ so $f'(2) = \frac{1}{2\sqrt{2-1}} = \frac{1}{2}$

$m_{tan} = \frac{1}{2}$ the point is (given) $(2,1)$

$y - 1 = \frac{1}{2}(x-2)$

$$y = \frac{1}{2}x - 1 + 1$$

$$\boxed{y = \frac{1}{2}x}$$



Question

Let $f(x) = 2x^2 + x$; determine $f'(x)$.

(a) $f'(x) = 4$

(b) $f'(x) = 2x + 1$

(c) $f'(x) = 4x + x$

(d) $f'(x) = 4x + 1$

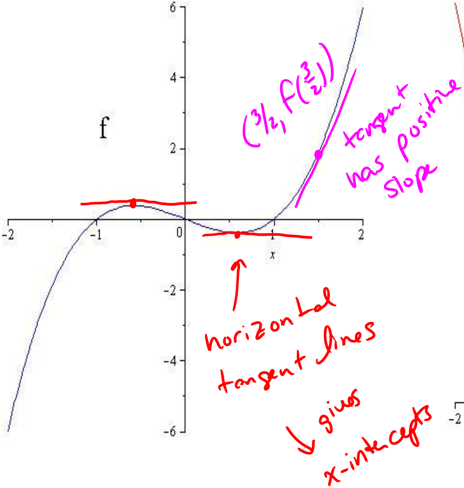
$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

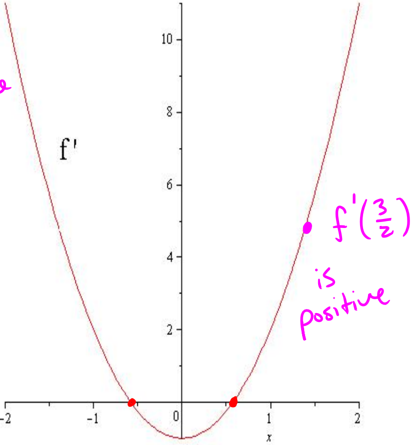
$$= \lim_{h \to 0} \frac{2(x+h)^2 + (x+h) - (2x^2 + x)}{h}$$

$$\vdots$$

$$= 4x + 1$$

How are the functions $f(x)$ and $f'(x)$ related?





Remarks:

- if $f(x)$ is a function of x, then $f'(x)$ is a new function of x (called the derivative of f)
- The number $f'(c)$ (if it exists) is the slope of the curve of $y = f(x)$ at the point $(c, f(c))$
- this is also the slope of the tangent line to the curve of y at $(c, f(c))$
- ”slope of the curve”, ”slope of the tangent line”, and ”rate of change” are the same concept

Definition: A function f is said to be *differentiable* at c if $f'(c)$ exists. It is called *differentiable* on an open interval I if it is differentiable at each point in I.

Failure to be Differentiable

We saw that the domain of $f(x) = \sqrt{x-1}$ is $[1, \infty)$ whereas the domain of its derivative $f'(x) = \frac{1}{2\sqrt{x-1}}$ was $(1, \infty)$. Hence f **is not differentiable at** 1**.**

An Example: Show that $y = |x|$ is not differentiable at zero. Let $f(x) = |x|$

If $f'(0)$ exists, it's equal to

$$\lim_{h \to 0} \frac{f(0+h) - f(0)}{h}$$

$$= \lim_{h \to 0} \frac{|0+h| - |0|}{h}$$

$$= \lim_{h \to 0} \frac{|h|}{h}$$

Recall $|h| = \begin{cases} h, & h \geq 0 \\ -h, & h < 0 \end{cases}$

We must take 2 one sided limits

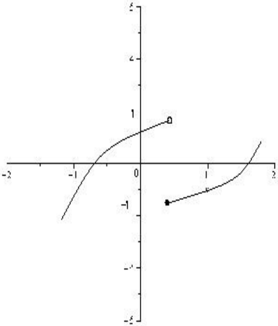
$$\lim_{h\to 0^-} \frac{|h|}{h} = \lim_{h\to 0^-} \frac{-h}{h} = \lim_{h\to 0^-} -1 = -1$$

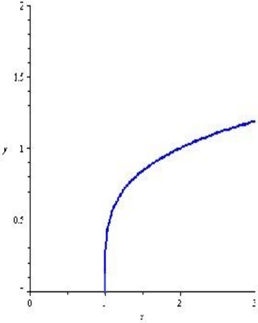
$$\lim_{h\to 0^+} \frac{|h|}{h} = \lim_{h\to 0^+} \frac{h}{h} = \lim_{h\to 0^+} 1 = 1$$

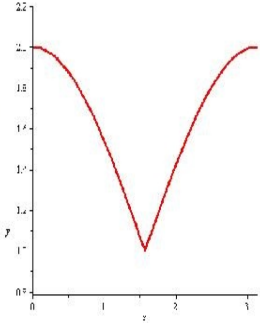
These disagree, hence $\lim_{h\to 0} \frac{|h|}{h}$ DNE

$f'(0)$ doesn't exist. That is, $y=|x|$ is <u>not</u> differentiable @ $x=0$.

Failure to be differentiable: Discontinuity, Vertical tangent, or Corner/Cusp







Corner

Cusp

Theorem

Differentiability implies continuity.

That is, if f is differentiable at c, then f is continuous at c. Note that the corner example shows that the converse of this is not true!

This means that a function can be continuous at a number c but not be differentiable there.

Questions

(1) **True or False:** Suppose that we know that $f'(3) = 2$. We can conclude that f is continuous at 3.

True, diff implies cont.

(2) **True or False:** Suppose that we know that $f'(1)$ does not exist. We can conclude that f is discontinuous at 1.

False, not necessarily – could have a corner or cusp.