Sept 14 Math 2306 sec. 53 Fall 2018

Section 6: Linear Equations Theory and Terminology

The context here is linear n^{th} order initial value problems. That is, solve

$$a_n(x)\frac{d^ny}{dx^n}+a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}}+\cdots+a_1(x)\frac{dy}{dx}+a_0(x)y=g(x)$$

subject to conditions

$$y(x_0)=y_0,\quad y'(x_0)=y_1,\quad \ldots,\quad y^{(n-1)}(x_0)=y_{n-1}.$$

The problem is called **homogeneous** if $g(x)\equiv 0$. Otherwise it is called **nonhomogeneous**.

First we focus on Homogeneous Equations

We'll consider the equation

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = 0$$

and assume that each a_i is continuous and a_n is never zero on the interval of interest.

Theorem:(the principle of superposition) If $y_1, y_2, \ldots, y_k$ are all solutions of this homogeneous equation on an interval I, then the *linear combination*

$$y(x) = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_k y_k(x)$$

is also a solution on I for any choice of constants $c_1, \ldots, c_k$.

Linear Dependence

Definition: A set of functions $f_1(x), f_2(x), \ldots, f_n(x)$ are said to be **linearly dependent** on an interval I if there exists a set of constants $c_1, c_2, \ldots, c_n$ with at least one of them being nonzero such that

$$c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x) = 0 \quad \text{for all} \quad x \text{ in } I.$$

A set of functions that is not linearly dependent on I is said to be **linearly independent** on I.

Example: A linearly Dependent Set

The functions $f_1(x) = \sin^2 x$, $f_2(x) = \cos^2 x$, and $f_3(x) = 1$ are linearly dependent on $I = (-\infty, \infty)$.

Recall $\sin^2 x + \cos^2 x = 1$ for all x

If we take $c_1 = 1$, $c_2 = 1$, $c_3 = -1$ then

$$c_1 f_1(x) + c_2 f_2(x) + c_3 f_3(x) =$$

$$\sin^2 x + \cos^2 x - 1 = 0$$

Note that at least one of the c's is nonzero.

Example: A linearly Independent Set

The functions $f_1(x) = \sin x$ and $f_2(x) = \cos x$ are linearly independent on $I = (-\infty, \infty)$.

Let's suppose that for some c_1, c_2 that

$$c_1 f_1(x) + c_2 f_2(x) = 0 \quad \text{for all } x.$$

Since this holds for all x, it holds when $x = 0$.

Note

$$c_1 f_1(0) + c_2 f_2(0) = 0$$

$$c_1 \sin(0) + c_2 \cos(0) = 0$$

$$c_1 \cdot 0 + c_2 \cdot 1 = 0 \implies c_2 = 0$$

Similarly, the equation has to hold when $x = \frac{\pi}{2}$. with $c_2 = 0$ already

$$c_1 f_1\left(\frac{\pi}{2}\right) = 0$$

$$c_1 \sin\left(\frac{\pi}{2}\right) = 0$$

$$c_1 \cdot 1 = 0 \quad \Rightarrow \quad c_1 = 0$$

Both c_1, c_2 <u>must</u> be zero.

Determine if the set is Linearly Dependent or Independent on $(-\infty, \infty)$

$$f_1(x) = x^2, \quad f_2(x) = 4x, \quad f_3(x) = x - x^2$$

Consider $\quad c_1 f_1(x) + c_2 f_2(x) + c_3 f_3(x) = 0 \quad$ for all x

for some c_1, c_2, c_3

$$c_1 x^2 + c_2 (4x) + c_3 (x - x^2) = 0$$

Collecting "like terms"

$$(c_1 - c_3) x^2 + (4c_2 + c_3) x = 0$$

This will be true if

$$c_1 - c_3 = 0 \quad \text{and} \quad 4c_2 + c_3 = 0$$

This will be true (for example) if

$$c_1 = 1, \quad c_3 = 1 \quad \text{and} \quad c_2 = \frac{-1}{4}$$

So $\quad f_1(x) - \frac{1}{4} f_2(x) + f_3(x) = 0 \quad$ for all x

The functions are linearly dependent.

Definition of Wronskian

Let $f_1, f_2, \ldots, f_n$ posses at least $n-1$ continuous derivatives on an interval I. The **Wronskian** of this set of functions is the determinant

$$W(f_1, f_2, \ldots, f_n)(x) = \begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ \vdots & \vdots & \vdots & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \cdots & f_n^{(n-1)} \end{vmatrix}.$$

(Note that, in general, this Wronskian is a function of the independent variable x.)

Determinants

If A is a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then its determinant

$$\det(A) = ad - bc.$$

If A is a 3×3 matrix $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, then its determinant

$$\det(A) = a_{11}\det\begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} - a_{12}\det\begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix} + a_{13}\det\begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$$

Determine the Wronskian of the Functions

$$f_1(x) = \sin x, \quad f_2(x) = \cos x$$

2 functions $\Rightarrow$

2×2 matrix

$$W(f_1,f_2)(x) = \begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix}$$

$$\begin{vmatrix} f_1(x) & f_2(x) \\ f_1'(x) & f_2'(x) \end{vmatrix}$$

$$= \sin x(-\sin x) - \cos x(\cos x)$$

$$= -\sin^2 x - \cos^2 x$$

$$= -(\sin^2 x + \cos^2 x) = -1$$

So $\quad W(f_1, f_2)(x) = -1$