September 14 Math 2306 sec. 56 Fall 2017

Section 5: First Order Equations Models and Applications

A Classic Mixing Problem A tank originally contains 500 gallons of pure water. Brine containing 2 pounds of salt per gallon is pumped in at a rate of 5gal/min. The well mixed solution is pumped out at the same rate. Find the amount of salt A(t) in pounds at the time t. Find the concentration of the mixture in the tank at t = 5 minutes.

A Classic Mixing Problem

From considering rate of increase and rate of decrease of the amount of salt at time *t*, we obtained the first order linear equation

$$\frac{dA}{dt} = r_i \cdot c_i - r_o \frac{A}{V} \implies \frac{dA}{dt} + \frac{r_o}{V} A = r_i c_i$$

where r_i is the rate of inflow of fluid, c_i is the concentration of salt in the incoming fluid, r_o is the rate of outflow of fluid, and V(t) is the volume of the mixture in the take. If the initial volume is V(0), then

$$V(t) = V(0) + (r_i - r_o)t.$$

If we know the starting amount of salt A(0), we can solve an IVP to find the amount of salt at all future times.

September 11, 2017

2/47

Solve the Mixing Problem

A tank originally contains 500 gallons of pure water. Brine containing 2 pounds of salt per gallon is pumped in at a rate of 5gal/min. The well mixed solution is pumped out at the same rate. Find the amount of salt A(t) in pounds at the time t. Find the concentration of the mixture in the tank at t = 5 minutes.

We determined

 $r_i = r_o = 5$ gal/min, $c_i = 2$ lb/gal, and A(0) = 0.

This gives us the IVP

$$\frac{dA}{dt} + \frac{5}{500}A = 10, \quad A(0) = 0$$

September 11, 2017

3/47

$$\frac{dA}{dt} + \frac{1}{100} A = 10 , A(0) = 0$$

$$15t \text{ order linear in should and form: } P(t) = \frac{1}{100}$$

$$Integrating \text{ factor } p = e = e = e^{-1} = e^{-1}$$

$$\frac{d}{dt} \left[e^{\frac{1}{100}t} A \right] = 10 e^{\frac{1}{100}t}$$

$$\int \frac{d}{dt} \left[e^{\frac{1}{100}t} A \right] dt = \int 10 e^{\frac{1}{100}t} dt$$

$$e^{\frac{1}{100}t} A = 10 (\frac{1}{100}) e^{\frac{1}{100}t} + C$$

< □ ▶ < □ ▶ < ■ ▶ < ■ ▶ < ■ ▶ E の へ ペ September 11, 2017 4/47

$$e^{\frac{1}{100}t} A = 1000e^{\frac{1}{100}t} + C$$

$$A = \frac{1000e^{\frac{1}{100}t} + C}{e^{\frac{1}{100}t}}$$

$$A = 1000 + Ce^{\frac{1}{100}t}$$

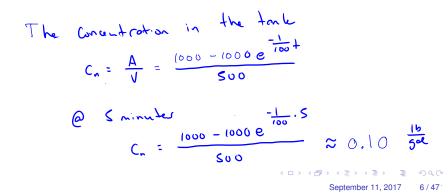
$$Using A(0) = 0$$

$$A(0) = 1000 + Ce^{\frac{1}{100}t} = 1000 + C = 1000$$

< □ ▶ < □ ▶ < ■ ▶ < ■ ▶ < ■ ▶ E の へ ペ September 11, 2017 5/47

$$C = -1000$$

The amount of salt A is
 $-\frac{1}{100}t$
A(t) = 1000 - 1000 e



$$r_i \neq r_o$$

Suppose that instead, the mixture is pumped out at 10 gal/min. Determine the differential equation satisfied by A(t) under this new condition.

Here
$$V(t) = V(0) + (r_i - r_0) t = 500 + (s - 10)t$$

$$C_0 = \frac{A}{V} = \frac{A}{500 - s t}$$

$$\frac{JA}{dt} + \frac{10}{500 - s t} A = 10$$

$$Volid$$

$$\frac{JA}{dt} + \frac{2}{100 - t} A = 10$$
for $0 < t < 100$

A Nonlinear Modeling Problem

A population P(t) of tilapia changes at a rate jointly proportional to the current population and the difference between the constant carrying capacity¹ M of the environment and the current population. Determine the differential equation satsified by P.

 $\frac{dP}{dt}$ a Population and M minus Population P M-P $\frac{dP}{dt} = k P(M-P)$

¹The carrying capacity is the maximum number of individuals that the environment can support due to limitation of space and resources.

Logistic Differential Equation

The equation

$$\frac{dP}{dt} = kP(M-P), \quad k, M > 0$$

is called a logistic growth equation.

Solve this equation² and show that for any $P(0) \neq 0$, $P \rightarrow M$ as $t \rightarrow \infty$.

The eqn is separable $\frac{1}{P(n-P)} \frac{dP}{dt} = k$ $\int \frac{1}{P(n-P)} dP = \int k dt$

²The partial fraction decomposition

$$\frac{1}{P(M-P)} = \frac{1}{M} \left(\frac{1}{P} + \frac{1}{M-P} \right)$$

is useful.

September 11, 2017 10 / 47

$$\int \frac{1}{m} \left(\frac{1}{p} + \frac{1}{m-p}\right) dP = \int kdt \qquad \text{multiput}$$

$$\int \left(\frac{1}{p} + \frac{1}{m-p}\right) dP = \int kMdt$$

$$\int \left(\frac{1}{p} + \frac{1}{m-p}\right) dP = kMt + C$$

$$\int \left|\frac{1}{m-p}\right| = kMt + C$$

$$\int \left|\frac{1}{m-p}\right| = kMt + C$$

$$\int \left|\frac{1}{m-p}\right| = e^{tMt} = e^{tMt}$$

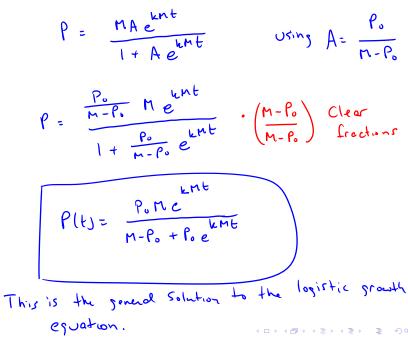
Let
$$A = e^{c}$$
 or $A = -e^{c}$ to obsorb on sign

$$\frac{P}{m-P} = A e^{kMt} \qquad \text{If } P(o) = P_{o}$$

$$\frac{P_{o}}{m-P_{o}} = A e^{o} = A \implies A = \frac{P_{o}}{m-P_{o}}$$
Solve $\frac{\Psi}{m}$ for P

Solve
$$\#$$
 for P
 $P = Ae^{knt} (n-P) = nAe^{-} APe^{-}$
 $P + APe^{kmt} = nAe^{knt}$
 $P (1 + Ae^{knt}) = nAe^{knt}$

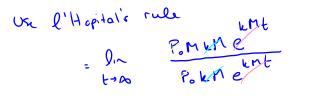
・ロト・西ト・モン・モー シック



September 11, 2017 13 / 47

Take
$$t \to \infty$$
 to see the Dong time population

$$\lim_{k \to \infty} P(k) = \lim_{k \to \infty} \frac{P_0 n e}{M_0 - P_0 + P_0 e^{knt}} = \frac{\infty}{\infty}$$



$$= \lim_{t \to \infty} M = M$$

September 11, 2017 14 / 47

<ロト <回 > < 回 > < 回 > < 回 > … 回

Section 6: Linear Equations Theory and Terminology

Recall that an nth order linear IVP consists of an equation

$$a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \dots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

to solve subject to conditions

$$y(x_0) = y_0, \quad y'(x_0) = y_1, \quad \dots, \quad y^{(n-1)}(x_0) = y_{n-1}.$$

The problem is called **homogeneous** if $g(x) \equiv 0$. Otherwise it is called nonhomogeneous.

> イロト 不得 トイヨト イヨト ヨー ろくの September 11, 2017

16/47

Theorem: Existence & Uniqueness

Theorem: If a_0, \ldots, a_n and g are continuous on an interval I, $a_n(x) \neq 0$ for each x in I, and x_0 is any point in I, then for any choice of constants y_0, \ldots, y_{n-1} , the IVP has a unique solution y(x) on I.

Put differently, we're guaranteed to have a solution exist, and it is the only one there is!

Homogeneous Equations

We'll consider the equation

$$a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \dots + a_1(x)\frac{dy}{dx} + a_0(x)y = 0$$

and assume that each a_i is continuous and a_n is never zero on the interval of interest.

Theorem: If y_1, y_2, \ldots, y_k are all solutions of this homogeneous equation on an interval *I*, then the *linear combination*

$$y(x) = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_k y_k(x)$$

September 11, 2017

18/47

is also a solution on I for any choice of constants c_1, \ldots, c_k .

This is called the **principle of superposition**.

Corollaries

- (i) If y_1 solves the homogeneous equation, the any constant multiple $y = cy_1$ is also a solution.
- (ii) The solution y = 0 (called the trivial solution) is always a solution to a homogeneous equation.

Big Questions:

- Does an equation have any **nontrivial** solution(s), and
- since y₁ and cy₁ aren't truly different solutions, what criteria will be used to call solutions distinct?

< □ > < □ > < □ > < ≡ > < ≡ > ≡
 September 11, 2017

19/47

4

Definition: A set of functions $f_1(x)$, $f_2(x)$, ..., $f_n(x)$ are said to be **linearly dependent** on an interval *I* if there exists a set of constants $c_1, c_2, ..., c_n$ with at least one of them being nonzero such that

$$c_1 f_1(x) + c_2 f_2(x) + \dots + c_n f_n(x) = 0$$
 for all x in I.

A set of functions that is not linearly dependent on *I* is said to be **linearly independent** on *I*.

Example: A linearly Dependent Set

The functions $f_1(x) = \sin^2 x$, $f_2(x) = \cos^2 x$, and $f_3(x) = 1$ are linearly dependent on $I = (-\infty, \infty)$.

Consider
$$C_1 f_1(x) + C_2 f_2(x) + C_3 f_3(x) = 0$$

 $C_1 S_{12} + C_2 (os^2 + C_3 + C_3) = 0$

Toke
$$C_1 = C_2 = 1$$
, $C_3 = -1$ (not all zero)
 $1 \le n^2 x + 1 \cdot C \le x - 1 \cdot 1 =$
 $(\le n^2 x + (\ o \ s^2 x) - 1 =$
 $1 = -1 = 0$

September 11, 2017 21 / 47

▲□▶ ▲□▶ ▲□▶ ▲□▶ ▲□ ● のへで

Example: A linearly Independent Set

The functions $f_1(x) = \sin x$ and $f_2(x) = \cos x$ are linearly independent on $I = (-\infty, \infty)$. Consider Ci fill + Cz fz(x) = O for all real X C, Sinx + C, Losx =0 If it's true for all x, it's true if x=0. C, Sin0 + C2 (000 = 0 $0 + C_2 = 0 \implies C_2 = 0$ It must ols hold if x= T/2. $c_1 \leq c_1 \leq c_2 \leq c_1 \leq c_2 \leq c_1 \leq c_2 \leq c_2 \leq c_1 \leq c_2 \leq c_2 \leq c_2 \leq c_1 \leq c_2 \leq c_2 \leq c_2 \leq c_1 \leq c_2 < c_2 \leq c_2 < c_2 \leq c_2 < c_2 \leq c_2 < c_2$ Ci, Ci much both be zerol September 11, 2017 22/47 Determine if the set is Linearly Dependent or $\mathcal{I} = (w, w)$ Independent $c, f, (x) + (c_2 f_2(x)) + (c_3 f_2(x)) = 0$

$$f_1(x) = x^2, \quad f_2(x) = 4x, \quad f_3(x) = x - x^2$$

 $C_{1} \chi^{2} + C_{2} (\eta_{\chi}) + C_{3} (\chi - \chi^{2}) = 0$

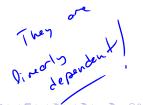
$$(c_1 - c_3) X^2 + (c_3 + 4c_2) X = 0$$

The $c_2 = c_3 = 1$, $c_2 = \frac{-1}{4}$ Not all zero

$$(1-1) X^{2} + (1+4) \left(\frac{-1}{4}\right) X =$$

0 + (1-1) X =

0+0 =0



Definition of Wronskian

Let $f_1, f_2, ..., f_n$ posses at least n - 1 continuous derivatives on an interval *I*. The **Wronskian** of this set of functions is the determinant

$$W(f_1, f_2, \dots, f_n)(x) = \begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ f'_1 & f'_2 & \cdots & f'_n \\ \vdots & \vdots & \vdots & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \cdots & f_n^{(n-1)} \end{vmatrix}$$

.

September 11, 2017

24/47

(Note that, in general, this Wronskian is a function of the independent variable x.)

Determinants

If *A* is a 2 × 2 matrix
$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$
, then its determinant $det(A) = ad - bc$.

If A is a 3 × 3 matrix
$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$
, then its determinant
$$det(A) = a_{11}det \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} - a_{12}det \begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix} + a_{13}det \begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$$

September 11, 2017 25 / 47

◆□> ◆圖> ◆理> ◆理> 三連

Determine the Wronskian of the Functions

$$f_{1}(x) = \sin x, \quad f_{2}(x) = \cos x$$

$$2x 2 \quad \sin k \quad \text{we have } 2 \text{ functions}$$

$$f_{1}'(x) = \cos x, \quad f_{2}'(x) = -\sin x$$

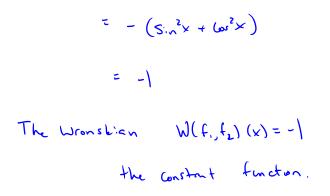
$$W(f_{1}, f_{2})(x) = \begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix}$$

$$= \sin x (-\sin x) - \cos x (\cos x)$$

$$= -\sin x (-\sin x) - \cos^{2} x$$

September 11, 2017 26 / 47

イロト イロト イヨト イヨト 二日



September 11, 2017 27 / 47