September 14 Math 2306 sec. 57 Fall 2017

Section 5: First Order Equations Models and Applications

A Classic Mixing Problem A tank originally contains 500 gallons of pure water. Brine containing 2 pounds of salt per gallon is pumped in at a rate of 5gal/min. The well mixed solution is pumped out at the same rate. Find the amount of salt $A(t)$ in pounds at the time t. Find the concentration of the mixture in the tank at $t = 5$ minutes.

A Classic Mixing Problem

From considering rate of increase and rate of decrease of the amount of salt at time t, we obtained the first order linear equation

$$\frac{dA}{dt} = r_i \cdot c_i - r_o \frac{A}{V} \quad \Longrightarrow \quad \frac{dA}{dt} + \frac{r_o}{V} A = r_i c_i,$$

where r_i is the rate of inflow of fluid, c_i is the concentration of salt in the incoming fluid, r_o is the rate of outflow of fluid, and $V(t)$ is the volume of the mixture in the take. If the initial volume is $V(0)$, then

$$V(t) = V(0) + (r_i - r_o)t.$$

If we know the starting amount of salt $A(0)$, we can solve an IVP to find the amount of salt at all future times.

Solve the Mixing Problem

A tank originally contains 500 gallons of pure water. Brine containing 2 pounds of salt per gallon is pumped in at a rate of 5gal/min. The well mixed solution is pumped out at the same rate. Find the amount of salt $A(t)$ in pounds at the time t. Find the concentration of the mixture in the tank at $t = 5$ minutes.

We determined

$$r_i = r_o = 5\,\text{gal/min}, \quad c_i = 2\,\text{lb/gal}, \quad \text{and} \quad A(0) = 0.$$

This gives us the IVP

$$\frac{dA}{dt} + \frac{5}{500}A = 10, \quad A(0) = 0.$$

Solve $\frac{dA}{dt} + \frac{1}{100} A = 10$, $A(0) = 0$

1^{st} order linear in standard form: $P(t) = \frac{1}{100}$

Get the integrating factor $\mu = e^{\int P(t)\,dt} = e^{\int \frac{1}{100}\,dt} = e^{\frac{1}{100}t}$

$$\frac{d}{dt}\left[e^{\frac{1}{100}t} A\right] = 10 e^{\frac{1}{100}t}$$

$$\int \frac{d}{dt}\left[e^{\frac{1}{100}t} A\right] dt = \int 10 e^{\frac{1}{100}t}\,dt$$

$$e^{\frac{1}{100}t} A = 10\left(\frac{1}{1/100}\right) e^{\frac{1}{100}t} + C$$

$$e^{\frac{1}{100}t} A = 1000\, e^{\frac{1}{100}t} + C$$

$$A = \frac{1000\, e^{\frac{1}{100}t} + C}{e^{\frac{1}{100}t}}$$

$$A(t) = 1000 + C e^{-\frac{1}{100}t}$$

Using $A(0)=0$

$$A(0) = 1000 + C e^{0} = 1000 + C = 0$$

$$C = -1000$$

So the amount of salt

$$\boxed{A = 1000 - 1000\, e^{-\frac{1}{100}t}}$$

The concentration C_n in the tank

$$C_n = \frac{A}{V}$$

At $t = 5$ min

$$C_n = \frac{A(5)}{V} = \frac{1000 - 1000\, e^{-\frac{1}{100}\cdot 5}}{500} \approx 0.1 \ \frac{\text{lb}}{\text{gal}}$$

$r_i \neq r_o$

Suppose that instead, the mixture is pumped out at 10 gal/min. Determine the differential equation satisfied by $A(t)$ under this new condition.

Here $V(t) = V(0) + (r_i - r_o)t = 500 + (5-10)t$

$$c_o = \frac{A}{V} = \frac{A}{500-5t}$$

$$\frac{dA}{dt} + \frac{10}{500-5t}A = 10$$

$$\frac{dA}{dt} + \frac{2}{100-t}A = 10$$

Valid for $0 < t < 100$

A Nonlinear Modeling Problem

A population $P(t)$ of tilapia changes at a rate jointly proportional to the current population and the difference between the constant carrying capacity[1] M of the environment and the current population. Determine the differential equation satsified by P.

$\frac{dP}{dt} \propto$ Population and difference between M and Population

P $\qquad$ $M-P$

$$\frac{dP}{dt} = kP(M-P)$$

[1] The carrying capacity is the maximum number of individuals that the environment can support due to limitation of space and resources.

Logistic Differential Equation

The equation

$$\frac{dP}{dt} = kP(M-P), \quad k, M > 0$$

is called a **logistic growth equation.**

Solve this equation[2] and show that for any $P(0) \neq 0$, $P \to M$ as $t \to \infty$.

Call $P(0) = P_0$

The ODE is separable

$$\frac{1}{P(M-P)} \frac{dP}{dt} = k \implies \int \frac{1}{P(M-P)}\,dP = \int k\,dt$$

[2] The partial fraction decomposition

$$\frac{1}{P(M-P)} = \frac{1}{M}\left(\frac{1}{P} + \frac{1}{M-P}\right)$$

is useful.

$$\int \frac{1}{M}\left(\frac{1}{P} + \frac{1}{M-P}\right) dP = \int k\,dt$$

mult. by M

$$\int \left(\frac{1}{P} + \frac{1}{M-P}\right) dP = \int kM\,dt$$

$$\ln|P| - \ln|M-P| = kMt + C$$

$$\ln\left|\frac{P}{M-P}\right| = kMt + C$$

$$\left|\frac{P}{M-P}\right| = e^{kMt+C} = e^{C} e^{kMt}$$

Letting $A = e^c$ or $A = -e^c$ to absorb any sign

$$* \quad \frac{P}{M-P} = A e^{kMt} \qquad \text{using } P(0) = P_0$$

$$\frac{P_0}{M-P_0} = Ae^0 \implies A = \frac{P_0}{M-P_0}$$

Solve * for P

$$P = Ae^{kMt}(M-P) = AMe^{kMt} - PAe^{kMt}$$

$$P + PAe^{kMt} = AMe^{kMt}$$

$$P(1 + Ae^{kMt}) = AMe^{kMt}$$

$$P = \frac{AM e^{kMt}}{1 + Ae^{kMt}}$$

Using $A = \frac{P_0}{M - P_0}$

$$P = \frac{\frac{P_0}{M-P_0} M e^{kMt}}{1 + \frac{P_0}{M-P_0} e^{kMt}} \cdot \left(\frac{M-P_0}{M-P_0}\right)$$ Clear fractions

$$P(t) = \frac{P_0 M e^{kMt}}{M - P_0 + P_0 e^{kMt}}$$

The general solution to the logistic equation is

$$P(t) = \frac{P_0 M e^{kMt}}{M - P_0 + P_0 e^{kMt}}$$

Take $t \to \infty$ to determine P in the long time.

$$\lim_{t\to\infty} P(t) = \lim_{t\to\infty} \frac{P_0 M e^{kMt}}{M - P_0 + P_0 e^{kMt}} = \text{"}\frac{\infty}{\infty}\text{"}$$

Use l'Hopital's rule

$$= \lim_{t\to\infty} \frac{\cancel{P_0} M \cancel{kM} \cancel{e^{kMt}}}{\cancel{P_0} \cancel{kM} \cancel{e^{kMt}}}$$

$$= \lim_{t\to\infty} M \qquad = M$$

Section 6: Linear Equations Theory and Terminology

Recall that an n^{th} order linear IVP consists of an equation

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

to solve subject to conditions

$$y(x_0) = y_0, \quad y'(x_0) = y_1, \quad \ldots, \quad y^{(n-1)}(x_0) = y_{n-1}.$$

The problem is called **homogeneous** if $g(x) \equiv 0$. Otherwise it is called **nonhomogeneous**.

Theorem: Existence & Uniqueness

Theorem: If $a_0, \ldots, a_n$ and g are continuous on an interval I, $a_n(x) \neq 0$ for each x in I, and x_0 is any point in I, then for any choice of constants $y_0, \ldots, y_{n-1}$, the IVP has a unique solution $y(x)$ on I.

Put differently, we're guaranteed to have a solution exist, and it is the only one there is!

Homogeneous Equations

We'll consider the equation

$$a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = 0$$

and assume that each a_i is continuous and a_n is never zero on the interval of interest.

Theorem: If $y_1, y_2, \ldots, y_k$ are all solutions of this homogeneous equation on an interval I, then the *linear combination*

$$y(x) = c_1y_1(x) + c_2y_2(x) + \cdots + c_ky_k(x)$$

is also a solution on I for any choice of constants $c_1, \ldots, c_k$.

This is called the **principle of superposition.**

Corollaries

(i) If y_1 solves the homogeneous equation, the any constant multiple $y = cy_1$ is also a solution.

(ii) The solution $y = 0$ (called the trivial solution) is always a solution to a homogeneous equation.

Big Questions:

- Does an equation have any **nontrivial** solution(s), and
- since y_1 and cy_1 aren't truly *different* solutions, what criteria will be used to call solutions distinct?

Linear Dependence

Definition: A set of functions $f_1(x), f_2(x), \ldots, f_n(x)$ are said to be **linearly dependent** on an interval I if there exists a set of constants $c_1, c_2, \ldots, c_n$ with at least one of them being nonzero such that

$$c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x) = 0 \quad \text{for all} \quad x \text{ in } I.$$

A set of functions that is not linearly dependent on I is said to be **linearly independent** on I.

If all c's must be zero to get all terms to cancel, the functions are linearly independent.

Example: A linearly Dependent Set

The functions $f_1(x) = \sin^2 x$, $f_2(x) = \cos^2 x$, and $f_3(x) = 1$ are linearly dependent on $I = (-\infty, \infty)$.

Consider $c_1 f_1(x) + c_2 f_2(x) + c_3 f_3(x) = 0$ for all x in $(-\infty, \infty)$

$$c_1 \sin^2 x + c_2 \cos^2 x + c_3 \cdot 1 = 0$$

Take $c_1 = c_2 = 1$, $c_3 = -1$ (not all zero)

$$1 \cdot \sin^2 x + 1 \cdot \cos^2 x + (-1) \cdot 1 =$$

$$\sin^2 x + \cos^2 x - 1 =$$

$$(\sin^2 x + \cos^2 x) - 1 =$$

$$1 - 1 = 0$$

Example: A linearly Independent Set

The functions $f_1(x) = \sin x$ and $f_2(x) = \cos x$ are linearly independent on $I = (-\infty, \infty)$.

Consider $c_1 f_1(x) + c_2 f_2(x) = 0$ for <u>all</u> x in $(-\infty, \infty)$

$$* \quad c_1 \sin x + c_2 \cos x = 0$$

If this holds for all x, it holds if $x = 0$.

$$c_1 \sin 0 + c_2 \cos 0 = 0 \Rightarrow c_2 \cdot 1 = 0 \Rightarrow c_2 = 0.$$

This holds when $x = \pi/2$ giving

$$c_1 \sin \pi/2 = 0 \Rightarrow c_1 \cdot 1 = 0 \Rightarrow c_1 = 0$$

$*$ only holds if $c_1 = c_2 = 0$.

Determine if the set is Linearly Dependent or Independent

$I = (-\infty, \infty)$

$$f_1(x) = x^2, \quad f_2(x) = 4x, \quad f_3(x) = x - x^2$$

$c_1 f_1(x) + c_2 f_2(x) + c_3 f_3(x) = 0$ for all x

$c_1 x^2 + c_2 (4x) + c_3 (x - x^2) = 0$

$(c_1 - c_3) x^2 + (c_3 + 4c_2) x = 0$

Consider $c_1 = c_3 = 4$ and $c_2 = -1$ ← not all zero

$(4-4) x^2 + (4 + 4(-1)) x =$
$0 + 0 = 0$

$\boxed{\text{They are linearly dependent!}}$

Definition of Wronskian

Let f_1, $f_2, \ldots, f_n$ posses at least $n-1$ continuous derivatives on an interval I. The **Wronskian** of this set of functions is the determinant

$$W(f_1, f_2, \ldots, f_n)(x) = \begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ \vdots & \vdots & \vdots & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \cdots & f_n^{(n-1)} \end{vmatrix}.$$

(Note that, in general, this Wronskian is a function of the independent variable x.)

Determinants

If A is a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then its determinant

$$\det(A) = ad - bc.$$

If A is a 3×3 matrix $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, then its determinant

$$\det(A) = a_{11}\det\begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} - a_{12}\det\begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix} + a_{13}\det\begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$$

Determine the Wronskian of the Functions

$$f_1(x) = \sin x, \quad f_2(x) = \cos x$$

2-functions $\Rightarrow$ 2×2 matrix

$f_1'(x) = \cos x$

$f_2'(x) = -\sin x$

$$W(f_1, f_2)(x) = \begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix}$$

$$= \sin x(-\sin x) - \cos x(\cos x)$$

$$= -\sin^2 x - \cos^2 x$$

$$= -\left(\sin^2 x + \cos^2 x\right)$$

$$= -1$$

So $W(f_1, f_2)(x) = -1$

for $f_1(x) = \sin x$, $f_2(x) = \cos x$