A Classic Mixing Problem A tank originally contains 500 gallons of pure water. Brine containing 2 pounds of salt per gallon is pumped in at a rate of 5gal/min. The well mixed solution is pumped out at the same rate. Find the amount of salt $A(t)$ in pounds at the time $t$. Find the concentration of the mixture in the tank at $t = 5$ minutes.
A Classic Mixing Problem

From considering rate of increase and rate of decrease of the amount of salt at time $t$, we obtained the first order linear equation

$$\frac{dA}{dt} = r_i \cdot c_i - r_o \frac{A}{V} \implies \frac{dA}{dt} + \frac{r_o}{V} A = r_i c_i,$$

where $r_i$ is the rate of inflow of fluid, $c_i$ is the concentration of salt in the incoming fluid, $r_o$ is the rate of outflow of fluid, and $V(t)$ is the volume of the mixture in the take. If the initial volume is $V(0)$, then

$$V(t) = V(0) + (r_i - r_o)t.$$

If we know the starting amount of salt $A(0)$, we can solve an IVP to find the amount of salt at all future times.
Solve the Mixing Problem

A tank originally contains 500 gallons of pure water. Brine containing 2 pounds of salt per gallon is pumped in at a rate of 5 gal/min. The well mixed solution is pumped out at the same rate. Find the amount of salt $A(t)$ in pounds at the time $t$. Find the concentration of the mixture in the tank at $t = 5$ minutes.

We determined

$$r_i = r_o = 5 \text{ gal/min}, \quad c_i = 2 \text{ lb/gal}, \quad \text{and} \quad A(0) = 0.$$ 

This gives us the IVP

$$\frac{dA}{dt} + \frac{5}{500} A = 10, \quad A(0) = 0.$$
Solve \( \frac{dA}{dt} + \frac{1}{100} A = 10 \), \( A(0) = 0 \)

1st order linear in standard form: \( \mu(t) = \frac{1}{100} \)

Get the integrating factor \( \mu = e^{\int \frac{1}{100} dt} = e^{\frac{1}{100} t} \)

\[ \frac{d}{dt} \left[ e^{\frac{1}{100} t} A \right] = 10 e^{\frac{1}{100} t} \]

\[ \int \frac{d}{dt} \left[ e^{\frac{1}{100} t} A \right] dt = \int 10 e^{\frac{1}{100} t} dt \]

\( e^{\frac{1}{100} t} A = 10 \left( \frac{1}{\frac{1}{100}} \right) e^{\frac{1}{100} t} + C \)
\[ e^{\frac{1}{100} t} A = 1000 e^{\frac{1}{100} t} + C \]

\[ A = \frac{1000 e^{\frac{1}{100} t} + C}{e^{\frac{1}{100} t}} \]

\[ A(t) = 1000 + C e^{-\frac{1}{100} t} \]

Using \( A(0) = 0 \)

\[ A(0) = 1000 + C e^{0} = 1000 + C = 0 \]

\[ C = -1000 \]
So the amount of salt

\[ A = 1000 - 1000 e^{\frac{-t}{100}} \]

The concentration \( C_n \) in the tank

\[ C_n = \frac{A}{V} \]

At \( t = 5 \) min

\[ C_n = \frac{A(5)}{V} = \frac{1000 - 1000 e^{\frac{-1}{100} \cdot 5}}{500} \approx 0.1 \text{ \ lb per gal} \]
$r_i \neq r_o$

Suppose that instead, the mixture is pumped out at 10 gal/min. Determine the differential equation satisfied by $A(t)$ under this new condition.

Here $V(t) = V(0) + (r_i - r_o)t = 500 + (5 - 10)t$

$$c_o = \frac{A}{V} = \frac{A}{500 - 5t}$$

$$\frac{dA}{dt} + \frac{10}{500 - 5t} A = 10 \quad \text{Valid for } 0 < t < 100$$

$$\frac{dA}{dt} + \frac{2}{100 - t} A = 10$$
A Nonlinear Modeling Problem

A population $P(t)$ of tilapia changes at a rate jointly proportional to the current population and the difference between the constant carrying capacity $M$ of the environment and the current population. Determine the differential equation satisfied by $P$.

$$\frac{dP}{dt} \propto \text{Population and difference between } M \text{ and Population} \Rightarrow M - P$$

$$\frac{dP}{dt} = kP(M - P)$$

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$^1$The carrying capacity is the maximum number of individuals that the environment can support due to limitation of space and resources.
Logistic Differential Equation

The equation

$$ \frac{dP}{dt} = kP(M - P), \quad k, M > 0 $$

is called a logistic growth equation.

Solve this equation\(^2\) and show that for any \(P(0) \neq 0\), \(P \to M\) as \(t \to \infty\).

\(^2\)The partial fraction decomposition

\[
\frac{1}{P(M - P)} = \frac{1}{M} \left( \frac{1}{P} + \frac{1}{M - P} \right)
\]

is useful.
\[ \int \frac{1}{M} \left( \frac{1}{\rho} + \frac{1}{M-\rho} \right) \, d\rho = \int k \, dt \]

\[ \int \left( \frac{1}{\rho} + \frac{1}{M-\rho} \right) \, d\rho = \int k M \, dt \]

\[ \ln|\rho| - \ln|M-\rho| = k M t + C \]

\[ \ln \left| \frac{\rho}{M-\rho} \right| = k M t + C \]

\[ \left| \frac{\rho}{M-\rho} \right| = e^{k M t + C} = e^C e^{k M t} \]
Letting $A = e^c$ or $A = -e^c$ to absorb any $\sinh$

\[ \frac{P}{m - \rho} = A e^{kmt} \quad \text{using } P(0) = P_0 \]

\[ \frac{P_0}{m - P_0} = A e^0 \quad \Rightarrow \quad A = \frac{P_0}{m - P_0} \]

Solve $*$ for $P$

\[ P = A e^{kmt} \quad (m - \rho) = A m e^{kmt} - P A e^{kmt} \]

\[ P + P A e^{kmt} = A m e^{kmt} \]

\[ P (1 + A e^{kmt}) = A m e^{kmt} \]
\[ P = \frac{A n e^{knt}}{1 + A e^{knt}} \]

Using \( A = \frac{P_0}{m - P_0} \)

\[ P = \frac{P_0}{m - P_0} \frac{m e^{knt}}{1 + \frac{P_0}{n - P_0} e^{knt}} \]

\[ P(t) = \frac{P_0 n e^{knt}}{m - P_0 + P_0 e^{knt}} \]
The general solution to the logistic equation is

\[ P(t) = \frac{P_0 M e^{knt}}{M - P_0 + P_0 e^{knt}} \]

Take \( t \to \infty \) to determine \( P \) in the long time.

\[
\lim_{t \to \infty} P(t) = \lim_{t \to \infty} \frac{P_0 M e^{knt}}{M - P_0 + P_0 e^{knt}} = \frac{\infty}{\infty}
\]

Use l'Hopital's rule.
\begin{align*}
\lim_{t \to \infty} \frac{P(t)^{k+1} e^t}{P(0)^{k+1} e^t} &= \lim_{t \to \infty} \frac{P(t)^{k+1}}{P(0)^{k+1}} e^{t-k} \\
&= \lim_{t \to \infty} M = M
\end{align*}
Recall that an $n^{th}$ order linear IVP consists of an equation

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

to solve subject to conditions

$$y(x_0) = y_0, \quad y'(x_0) = y_1, \quad \ldots, \quad y^{(n-1)}(x_0) = y_{n-1}.$$ 

The problem is called **homogeneous** if $g(x) \equiv 0$. Otherwise it is called **nonhomogeneous**.
Theorem: Existence & Uniqueness

**Theorem:** If $a_0, \ldots, a_n$ and $g$ are continuous on an interval $I$, $a_n(x) \neq 0$ for each $x$ in $I$, and $x_0$ is any point in $I$, then for any choice of constants $y_0, \ldots, y_{n-1}$, the IVP has a unique solution $y(x)$ on $I$.

Put differently, we’re guaranteed to have a solution exist, and it is the only one there is!
Homogeneous Equations

We’ll consider the equation

\[ a_n(x) \frac{d^n y}{dx^n} + a_{n-1}(x) \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x) \frac{dy}{dx} + a_0(x)y = 0 \]

and assume that each \( a_i \) is continuous and \( a_n \) is never zero on the interval of interest.

**Theorem:** If \( y_1, y_2, \ldots, y_k \) are all solutions of this homogeneous equation on an interval \( I \), then the linear combination

\[ y(x) = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_k y_k(x) \]

is also a solution on \( I \) for any choice of constants \( c_1, \ldots, c_k \).

This is called the **principle of superposition**.
Corollaries

(i) If $y_1$ solves the homogeneous equation, the any constant multiple $y = cy_1$ is also a solution.

(ii) The solution $y = 0$ (called the trivial solution) is always a solution to a homogeneous equation.

Big Questions:

▶ Does an equation have any nontrivial solution(s), and
▶ since $y_1$ and $cy_1$ aren’t truly different solutions, what criteria will be used to call solutions distinct?
Linear Dependence

**Definition:** A set of functions $f_1(x), f_2(x), \ldots, f_n(x)$ are said to be **linearly dependent** on an interval $I$ if there exists a set of constants $c_1, c_2, \ldots, c_n$ with at least one of them being nonzero such that

$$c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x) = 0 \quad \text{for all } x \text{ in } I.$$

A set of functions that is not linearly dependent on $I$ is said to be **linearly independent** on $I$.

If all $c$'s must be zero to get all terms to cancel, the functions are linearly independent.
Example: A linearly Dependent Set

The functions \( f_1(x) = \sin^2 x \), \( f_2(x) = \cos^2 x \), and \( f_3(x) = 1 \) are linearly dependent on \( I = (-\infty, \infty) \).

Consider \( c_1 f_1(x) + c_2 f_2(x) + c_3 f_3(x) = 0 \) \( \forall x \in (-\infty, \infty) \)

\[
c_1 \sin^2 x + c_2 \cos^2 x + c_3 \cdot 1 = 0
\]

Take \( c_1 = c_2 = 1 \), \( c_3 = -1 \) \( \text{(not all zero)} \)

\[
1 \cdot \sin^2 x + 1 \cdot \cos^2 x + (-1) \cdot 1 = \\
\sin^2 x + \cos^2 x - 1 = \\
(\sin^2 x + \cos^2 x) - 1 = \\
1 - 1 = 0
\]
Example: A linearly Independent Set

The functions $f_1(x) = \sin x$ and $f_2(x) = \cos x$ are linearly independent on $I = (-\infty, \infty)$.

Consider $c_1f_1(x) + c_2f_2(x) = 0$ for all $x$ in $(-\infty, \infty)$.

\[ * \quad c_1 \sin x + c_2 \cos x = 0 \]

If this holds for all $x$, it holds if $x = 0$.

\[ c_1 \sin 0 + c_2 \cos 0 = 0 \Rightarrow c_2 \cdot 1 = 0 \Rightarrow c_2 = 0. \]

This holds when $x = \pi/2$ giving

\[ c_1 \sin \pi/2 = 0 \Rightarrow c_1 \cdot 1 = 0 \Rightarrow c_1 = 0 \]

\[ * \quad \text{only holds if } c_1 = c_2 = 0. \]
Determine if the set is Linearly Dependent or Independent

\[ I = (-\infty, \infty) \]

\[ f_1(x) = x^2, \quad f_2(x) = 4x, \quad f_3(x) = x - x^2 \]

\[ c_1 f_1(x) + c_2 f_2(x) + c_3 f_3(x) = 0 \quad \text{for all } x \]

\[ c_1 x^2 + c_2 (4x) + c_3 (x - x^2) = 0 \]

\[ (c_1 - c_3) x^2 + (c_3 + 4c_2) x = 0 \]

Consider \( c_1 = c_3 = 4 \) and \( c_1 = -1 \) \( \leftarrow \) not all zero

\[ (4-4) x^2 + (4+4(-1)) x = 0 \]

\[ 0 + 0 = 0 \]

They are linearly dependent!
Definition of Wronskian

Let \( f_1, f_2, \ldots, f_n \) posses at least \( n - 1 \) continuous derivatives on an interval \( I \). The **Wronskian** of this set of functions is the determinant

\[
W(f_1, f_2, \ldots, f_n)(x) = \begin{vmatrix}
  f_1 & f_2 & \cdots & f_n \\
  f'_1 & f'_2 & \cdots & f'_n \\
  \vdots & \vdots & \ddots & \vdots \\
  f_1^{(n-1)} & f_2^{(n-1)} & \cdots & f_n^{(n-1)}
\end{vmatrix}.
\]

(Note that, in general, this Wronskian is a function of the independent variable \( x \).)
Determinants

If $A$ is a $2 \times 2$ matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then its determinant

$$\text{det}(A) = ad - bc.$$ 

If $A$ is a $3 \times 3$ matrix $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, then its determinant

$$\text{det}(A) = a_{11}\text{det} \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} - a_{12}\text{det} \begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix} + a_{13}\text{det} \begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}.$$
Determine the Wronskian of the Functions

\[ f_1(x) = \sin x, \quad f_2(x) = \cos x \]

2 functions \( \Rightarrow \) 2x2 matrix

\[
W(f_1, f_2)(x) = \begin{vmatrix}
\sin x & \cos x \\
\cos x & -\sin x \\
\end{vmatrix}
\]

\[
= \sin x (-\sin x) - \cos x (\cos x)
\]

\[
= -\sin^2 x - \cos^2 x
\]
\[ = - (\sin^2 x + \cos^2 x) \]
\[ = -1 \]
So \[ W(f_1, f_2)(x) = -1 \]
for \( f_1(x) = \sin x \), \( f_2(x) = \cos x \).