September 14 Math 3260 sec. 58 Fall 2017

Section 2.2: Inverse of a Matrix

If A is an $n \times n$ matrix, we seek a matrix A^{-1} that satisfies the condition

$$A^{-1}A = AA^{-1} = I_n.$$

If such matrix A^{-1} exists, we'll say that A is **nonsingular** (a.k.a. *invertible*). Otherwise, we'll say that A is **singular**.

Theorem (2×2 case)

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. If $ad - bc \neq 0$, then A is invertible and

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

If $ad - bc = 0$, then A is singular.

The quantity $ad - bc$ is called the **determinant** of A.

Theorem: If A is an invertible $n \times n$ matrix, then for each $\mathbf{b}$ in $\mathbb{R}^n$, the equation $A\mathbf{x} = \mathbf{b}$ has unique solution $\mathbf{x} = A^{-1}\mathbf{b}$.

Example

Solve the system using a matrix inverse

$$\begin{array}{rcrcr} 4x_1 & + & x_2 & = & 7 \\ -2x_1 & + & 3x_2 & = & 7 \end{array}$$

Theorem

(i) If A is invertible, then A^{-1} is also invertible and

$$\left(A^{-1}\right)^{-1} = A.$$

(ii) If A and B are invertible $n \times n$ matrices, then the product AB is also invertible[1] with

$$(AB)^{-1} = B^{-1}A^{-1}.$$

(iii) If A is invertible, then so is A^T. Moreover

$$\left(A^T\right)^{-1} = \left(A^{-1}\right)^T.$$

[1] This can generalize to the product of k invertible matrices.

Elementary Matrices

Definition: An **elementary** matrix is a square matrix obtained from the identity by performing one elementary row operation.

Examples:

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}, \quad E_3 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Action of Elementary Matrices

Let $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$, and compute the following products

$$E_1 A, \quad E_2 A, \quad \text{and} \quad E_3 A.$$

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$$

$$E_3 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Remarks

- Elementary row operations can be equated with matrix multiplication (multiply on the left by an elementary matrix),
- Each elementary matrix is invertible where the inverse *undoes* the row operation,
- Reduction to rref is a sequence of row operations, so it is a sequence of matrix multiplications

$$\mathrm{rref}(A) = E_k \cdots E_2 E_1 A.$$

Theorem

An $n \times n$ matrix A is invertible if and only if it is row equivalent to the identity matrix I_n. Moreover, if

$$\mathrm{rref}(A) = E_k \cdots E_2 E_1 A = I_n, \quad \text{then} \quad A = (E_k \cdots E_2 E_1)^{-1} I_n.$$

That is,

$$A^{-1} = \left[(E_k \cdots E_2 E_1)^{-1} \right]^{-1} = E_k \cdots E_2 E_1.$$

The sequence of operations that reduces A to I_n, transforms I_n into A^{-1}.

This last observation—operations that take A to I_n also take I_n to A^{-1}—gives us a method for computing an inverse!

Algorithm for finding A^{-1}

To find the inverse of a given matrix A:

- Form the $n \times 2n$ augmented matrix $[A \quad I]$.
- Perform whatever row operations are needed to get the first n columns (the A part) to rref.
- If rref(A) is I, then $[A \quad I]$ is row equivalent to $[I \quad A^{-1}]$, and the inverse A^{-1} will be the last n columns of the reduced matrix.
- If rref(A) is NOT I, then A is not invertible.

Remarks: We don't need to know ahead of time if A is invertible to use this algorithm.
If A is singular, we can stop as soon as it's clear that rref$(A) \neq I$.

Examples: Find the Inverse if Possible

(a) $\begin{bmatrix} 1 & 2 & -1 \\ -4 & -7 & 3 \\ -2 & -6 & 4 \end{bmatrix}$

Examples: Find the Inverse if Possible

(b) $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 5 & 6 & 0 \end{bmatrix}$

Solve the linear system if possible

$$\begin{array}{rcrcrcr} x_1 & + & 2x_2 & + & 3x_3 & = & 3 \\ & & x_2 & + & 4x_3 & = & 3 \\ 5x_1 & + & 6x_2 & & & = & 4 \end{array}$$

Section 2.3: Characterization of Invertible Matrices

Given an $n \times n$ matrix A, we can think of

- A matrix equation $A\mathbf{x} = \mathbf{b}$;
- A linear system that has A as its coefficient matrix;
- A linear transformation $T : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ defined by $T(\mathbf{x}) = A\mathbf{x}$;
- Not to mention things like its **pivots**, its **rref**, the linear dependence/independence of its columns, blah blah blah...

Question: How is this stuff related, and how does being singular or invertible tie in?

Theorem: Suppose A is $n \times n$. The following are equivalent. [2]

(a) A is invertible.
(b) A is row equivalent to I_n.
(c) A has n pivot positions.
(d) $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.
(e) The columns of A are linearly independent.
(f) The transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one to one.
(g) $A\mathbf{x} = \mathbf{b}$ is consistent for every $\mathbf{b}$ in $\mathbb{R}^n$.
(h) The columns of A span $\mathbb{R}^n$.
(i) The transformation $\mathbf{x} \mapsto A\mathbf{x}$ is onto.
(j) There exists an $n \times n$ matrix C such that $CA = I$.
(k) There exists an $n \times n$ matrix D such that $AD = I$.
(l) A^T is invertible.

[2] Meaning all are true or none are true.

Theorem: (An inverse matrix is unique.)

Let A and B be $n \times n$ matrices. If $AB = I$, then A and B are both invertible with $A^{-1} = B$ and $B^{-1} = A$.

Invertible Linear Transformations

Definition: A linear transformation $T : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is said to be **invertible** if there exists a function $S : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ such that both

$$S(T(\mathbf{x})) = \mathbf{x} \quad \text{and} \quad T(S(\mathbf{x})) = \mathbf{x}$$

for every $\mathbf{x}$ in $\mathbb{R}^n$.

If such a function exists, we typically denote it by

$$S = T^{-1}.$$

Theorem (Invertibility of a linear transformation and its matrix)

Let $T:\mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a linear transformation and A its standard matrix. Then T is invertible if and only if A is invertible. Moreover, if T is invertible, then

$$T^{-1}(\mathbf{x}) = A^{-1}\mathbf{x}$$

for every $\mathbf{x}$ in $\mathbb{R}^n$.

Example

Use the standard matrix to determine if the linear transformation is invertible. If it is invertible, characterize the inverse transformation.

$$T:\mathbb{R}^2\longrightarrow\mathbb{R}^2,\quad \text{given by}\quad T(x_1,x_2)=(3x_1-x_2,4x_2).$$

Example

Suppose $T : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a one to one linear transformation. Can we determine whether T is onto? Why (or why not)?