Sept. 16 Math 1190 sec. 51 Fall 2016

Section 2.3: The Derivative of a Polynomial; The Derivative of e^x

First some notation: If $y = f(x)$, the following notation are interchangeable:

$$f'(x) = y'(x) = y' = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx}f(x) = Df(x) = D_x f(x)$$

Leibniz Notation: $$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx}$$

You can think of $D,$ or $\dfrac{d}{dx}$ as an "operator."

It acts on a function to produce a new function—its derivative.
Taking a derivative is referred to as *differentiation*.

Some Derivative Rules

The derivative of a constant function is zero.

$$\frac{d}{dx} c = 0$$

The derivative of the identity function is one.

$$\frac{d}{dx} x = 1$$

For positive integer n^*,

$$\frac{d}{dx} x^n = n x^{n-1}$$

This last one is called the **power rule**.

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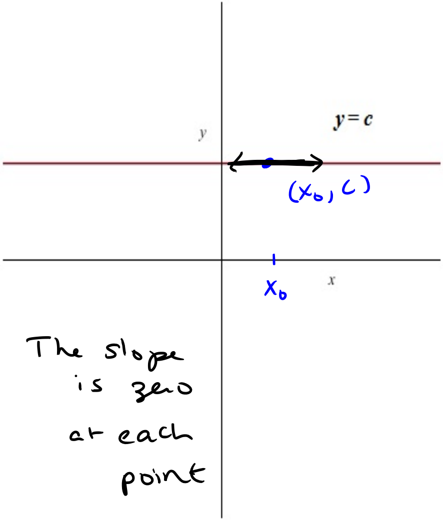
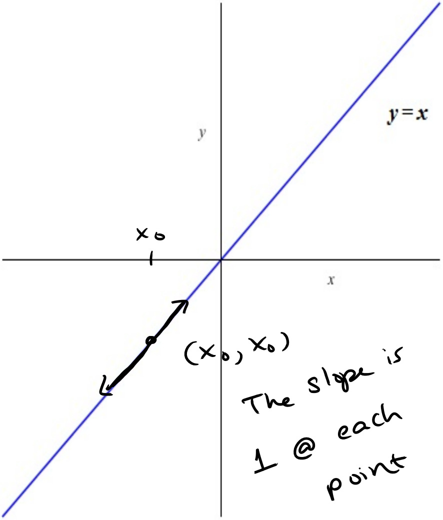
*This rule turns out to hold for any real number n, though the proofs for more general cases require results yet to come.

$$\frac{d}{dx}c = 0, \qquad \frac{d}{dx}x = 1$$



The slope is zero at each point



The slope is 1 @ each point

Evaluate Each Derivative

(a) $\dfrac{d}{dx}(-7) =$ 0 Constant

(b) $\dfrac{d}{dx} 3\pi =$ 0 constant

(c) $\dfrac{d}{dx} x^9 =$ $9x^{9-1} = 9x^8$ power rule

More Derivative Rules

Assume f and g are differentiable functions and k is a constant.

Constant multiple rule: $$\frac{d}{dx}kf(x) = k\frac{d}{dx}f(x)$$

Sum rule: $$\frac{d}{dx}(f(x)+g(x)) = \frac{d}{dx}f(x)+\frac{d}{dx}g(x)$$

Difference rule: $$\frac{d}{dx}(f(x)-g(x)) = \frac{d}{dx}f(x)-\frac{d}{dx}g(x)$$

The rules we have thus far allow us to find the derivative of any polynomial function.

Example: Evaluate Each Derivative

(a) $\frac{d}{dx}(x^4 - 3x^2) = \frac{d}{dx} x^4 - \frac{d}{dx}(3x^2)$ Break up difference

$= \frac{d}{dx} x^4 - 3 \frac{d}{dx} x^2$ factor out constant

$= 4x^{4-1} - 3(2x^{2-1})$ use power rule

$= 4x^3 - 6x$

(b) $$\frac{d}{dx}(2x^3+3x^2-12x+1) = \frac{d}{dx}2x^3 + \frac{d}{dx}3x^2 - \frac{d}{dx}12x + \frac{d}{dx}1$$

$$= 2\frac{d}{dx}x^3 + 3\frac{d}{dx}x^2 - 12\frac{d}{dx}x + \frac{d}{dx}1$$

$$= 2(3x^2) + 3(2x) - 12 \cdot 1 + 0$$

$$= 6x^2 + 6x - 12$$

$$= 6(x^2 + x - 2)$$

Example

If $f(x) = 2x^3 + 3x^2 - 12x + 1$, find all points on the graph of f at which the slope of the graph is zero.

We know that $f'(x) = 6(x^2 + x - 2)$

At the point $(c, f(c))$. The slope of the tangent line is $f'(c)$.

So we need to solve*

$f'(c) = 0$ for c

(Horizontal line's have slope zero)

$$f'(c) = 6(c^2 + c - 2) = 0$$

$$c^2 + c - 2 = 0$$

$$(c+2)(c-1) = 0 \Rightarrow c = -2 \text{ or } c = 1$$

We need the y-values

$$f(1) = 2 \cdot 1^3 + 3 \cdot 1^2 - 12 \cdot 1 + 1 = -6$$

$$f(-2) = 2(-2)^3 + 3(-2)^2 - 12(-2) + 1 = 21$$

The points are

$(1, -6)$ and $(-2, 21)$

The Derivative of e^x

Consider $a > 0$ and $a \neq 1$. Let $f(x) = a^x$. Analyze the limit $f'(0)$ and $f'(x)$

If the limit exists (it does - take my word)

Then

$$f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h}$$

$$= \lim_{h \to 0} \frac{a^{0+h} - a^0}{h}$$

$$= \lim_{h \to 0} \frac{a^h - 1}{h}$$

$$\boxed{f'(0) = \lim_{h \to 0} \frac{a^h - 1}{h} \quad \text{which is some number}}$$

Okay, so

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{a^{x+h} - a^x}{h}$$

$$= \lim_{h \to 0} \frac{(a^h - 1)a^x}{h}$$

$$= \left(\lim_{h \to 0} \frac{a^h - 1}{h} \right) a^x$$

$$= f'(0)\, a^x$$

$$a^{x+h} = a^x \cdot a^h$$

$$a^{x+h} - a^x =$$

$$(a^h - 1)\, a^x$$

So if $f(x) = a^x$ then

$$f'(x) = f'(0)\, a^x = f'(0)\, f(x)$$

$f'(x)$ is a "constant times a^x"

i.e. a "constant times $f(x)$"

The Derivative of e^x

Definition: The number e is defined[†] by the property

$$\lim_{h \to 0} \frac{e^h - 1}{h} = 1.$$

It follows that

Theorem: $y = e^x$ is differentiable (at all real numbers) and

$$\frac{d}{dx} e^x = e^x.$$

[†]This is one of several mutually consistent ways to defined this number. Numerically, $e \approx 2.718282$.

Section 2.4: Differentiating a Product or Quotient; Higher Order Derivatives

Motivating Example: Evaluate the derivative

$$\frac{d}{dx}[x^3(2x^2-6x+17)] = \frac{d}{dx}\left(2x^5-6x^4+17x^3\right)$$

$$= 2(5x^4)-6(4x^3)+17(3x^2)$$

$$= 10x^4-24x^3+51x^2$$

we just do the algebra first

Derivative of A Product

Now consider evaluating the derivative

$$\frac{d}{dx}[(3x^5-2x^2+x)(x^3-2x^2+x-1)]$$

We could, in principle, do the same here. Do the algebra first, then take the derivative.

Fortunately, there is an alternative.

Derivative of A Product

Theorem: (Product Rule) Let f and g be differentiable functions of x. Then the product $f(x)g(x)$ is differentiable. Moreover

$$\frac{d}{dx}[f(x)g(x)] = f'(x)g(x) + f(x)g'(x).$$

This can be stated using Leibniz notation as

$$\frac{d}{dx}[f(x)g(x)] = \frac{df}{dx}g(x) + f(x)\frac{dg}{dx}.$$

Example

Compute $\frac{d}{dx}x^5$ using the product rule with $f(x) = x^2$ and $g(x) = x^3$. Compare this with the result from the power rule on x^5.

$\frac{d}{dx} x^5 = 5x^4$ by the power rule.

$$\frac{d}{dx} x^5 = \frac{d}{dx} x^2 \cdot x^3 = \left(\frac{d}{dx} x^2\right) x^3 + x^2 \left(\frac{d}{dx} x^3\right)$$

$$= 2x \cdot x^3 + x^2 (3x^2)$$

$$= 2x^4 + 3x^4 = 5x^4$$

as expected.

Example

Evaluate $\dfrac{d}{dx}[(3x^5-2x^2+x)(x^3-2x^2+x-1)]$

Let $f(x) = 3x^5 - 2x^2 + x$

$f'(x) = 15x^4 - 4x + 1$

$g(x) = x^3 - 2x^2 + x - 1$

$g'(x) = 3x^2 - 4x + 1$

$$\frac{d}{dx}\left[(3x^5-2x^2+x)(x^3-2x^2+x-1)\right]$$

$$=(15x^2-4x+1)(x^3-2x^2+x-1)+(3x^5-2x^2+1)(3x^2-4x+1)$$

Example

Evaluate $\frac{d}{dx}e^{2x}$ using the product rule.

$$e^{2x} = e^{x} \cdot e^{x} \quad \text{so}$$

$$\frac{d}{dx}e^{2x} = \frac{d}{dx}\left(e^{x}\,e^{x}\right) = \left(\frac{d}{dx}e^{x}\right)e^{x} + e^{x}\left(\frac{d}{dx}e^{x}\right)$$

$$= e^{x} \cdot e^{x} + e^{x} \cdot e^{x}$$

$$= e^{2x} + e^{2x}$$

$$= 2e^{2x}$$