September 16 Math 2306 sec 51 Fall 2015

Section 4.1 Some Theory of Linear Equations

Definition of Wronskian Let f_1, $f_2, \ldots, f_n$ posses at least $n-1$ continuous derivatives on an interval I. The **Wronskian** of this set of functions is the determinant

$$W(f_1, f_2, \ldots, f_n)(x) = \begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ \vdots & \vdots & \vdots & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \cdots & f_n^{(n-1)} \end{vmatrix}.$$

Determine the Wronskian of the Functions

$$f_1(x) = xe^x, \quad f_2(x) = e^x$$

$$f_1'(x) = e^x + xe^x, \quad f_2'(x) = e^x$$

$$W(f_1, f_2)(x) = \begin{vmatrix} xe^x & e^x \\ e^x + xe^x & e^x \end{vmatrix}$$

$$= xe^x(e^x) - (e^x + xe^x)e^x$$

$$= xe^{2x} - \left(e^{2x} + xe^{2x}\right)$$

$$= xe^{2x} - e^{2x} - xe^{2x} \qquad = -e^{2x}$$

An Observation

The set $\{\sin x, \cos x\}$ is linearly independent on $(-\infty, \infty)$ and we found that

$$W(\sin x, \cos x)(x) = -1.$$

The set $\{x^2, 4x, x - x^2\}$ is linearly dependent on $(-\infty, \infty)$ and we found that

$$W(x^2, 4x, x - x^2)(x) = 0.$$

Theorem (a test for linear independence)

Let $f_1, f_2, \ldots, f_n$ be $n-1$ times continuously differentiable on an interval I. If there exists x_0 in I such that $W(f_1, f_2, \ldots, f_n)(x_0) \neq 0$, then the functions are **linearly independent** on I.

If $y_1, y_2, \ldots, y_n$ are n solutions of the linear homogeneous n^{th} order equation on an interval I, then the solutions are **linearly independent** on I if and only if $W(y_1, y_2, \ldots, y_n)(x) \neq 0$ for[1] each x in I.

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[1] For solutions of one linear homogeneous ODE, the Wronskian is either always zero or is never zero.

Determine if the functions are linearly dependent or independent:

$y_1 = e^x, \quad y_2 = e^{-2x} \quad I = (-\infty, \infty)$

$y_1' = e^x$

$y_2' = -2e^{-2x}$

Use the wronskian:

$$W(y_1, y_2)(x) = \begin{vmatrix} e^x & e^{-2x} \\ e^x & -2e^{-2x} \end{vmatrix} = e^x(-2e^{-2x}) - e^x(e^{-2x})$$

$$= -2e^{-x} - e^{-x} = -3e^{-x}$$

$$W(e^{x}, e^{-2x})(x) = -3e^{-x} \neq 0 \quad \text{for all } x$$

Hence y_1, y_2 are linearly independent.

Fundamental Solution Set

We're still considering this equation

$$a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = 0$$

with the assumptions $a_n(x) \neq 0$ and $a_i(x)$ are continuous on I.

Definition: A set of functions $y_1, y_2, \dots, y_n$ is a **fundamental solution set** of the n^{th} order homogeneous equation provided they

(i) are solutions of the equation,

(ii) there are n of them, and

(iii) they are linearly independent.

Theorem: Under the assumed conditions, the equation has a fundamental solution set.

General Solution of n^{th} order Linear Homogeneous Equation

Let $y_1, y_2, \dots, y_n$ be a fundamental solution set of the n^{th} order linear homogeneous equation. Then the **general solution** of the equation is

$$y(x) = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x),$$

where $c_1, c_2, \dots, c_n$ are arbitrary constants.

Example

Verify that $y_1 = e^x$ and $y_2 = e^{-x}$ form a fundamental solution set of the ODE

$$y'' - y = 0 \quad \text{on} \quad (-\infty, \infty),$$

and determine the general solution.

The eqn is 2^{nd} order; there are 2 functions, so the condition (ii) is satisfied.

Verify they solve the o.d.e.

$$y_1 = e^x, \quad y_1'(x) = e^x, \quad y_1''(x) = e^x$$

$$y_1'' - y_1 = e^x - e^x = 0 \quad \text{as required.}$$

$y_2(x) = e^{-x}$, $y_2'(x) = -e^{-x}$, $y_2''(x) = e^{-x}$

$$y_2'' - y_2 = e^{-x} - e^{-x} = 0$$ again as required.

Both y_1 and y_2 solve the ODE, so condition (i) is satisfied.

Check for linear independence:

$$W(y_1, y_2)(x) = \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix} = e^x(-e^{-x}) - e^x(e^{-x}) =$$

$$= -1 - 1 = -2 \neq 0$$

Since $W(y_1, y_2)(x) \neq 0$, y_1 and y_2 are linearly independent. Condition (iii) is satisfied.

Hence y_1, y_2 form a fundamental solution set.

The general solution is $y(x) = C_1 y_1(x) + C_2 y_2(x)$

i.e. $y(x) = C_1 e^{x} + C_2 e^{-x}$.

Consider $x^2y'' - 4xy' + 6y = 0$ for $x > 0$

2nd order equation

Determine which if any of the following sets of functions is a fundamental solution set.

(a) $y_1 = 2x^2, \quad y_2 = x^2$ ← not possible, linearly dependent
$y_1(x) - 2y_2(x) = 0$ for all $x > 0$

(b) $y_1 = x^2, \quad y_2 = x^{-2}$ ← y_2 doesn't solve the DE (see below)

(c) $y_1 = x^3, \quad y_2 = x^2$

(d) $y_1 = x^2, \quad y_2 = x^3, \quad y_3 = x^{-2}$ ← not possible, too many functions

Check option (b)

$$y_2(x) = x^{-2}\,,\quad y_2'(x) = -2x^{-3}\,,\quad y_2''(x) = 6x^{-4}$$

$$x^2 y_2'' - 4x y_2' + 6y_2 = x^2(6x^{-4}) - 4x(-2x^{-3}) + 6(x^{-2})$$

$$= 6x^{-2} + 8x^{-2} + 6x^{-2} = 20x^{-2} \neq 0$$

not a solution

Check option (c)

$$y_1 = x^3\,,\quad y_1' = 3x^2\,,\quad y_1'' = 6x$$

$$x^2 y_1'' - 4x y_1' + 6y_1 = x^2(6x) - 4x(3x^2) + 6x^3$$

$$= 6x^3 - 12x^3 + 6x^3 = 0$$

$y_2 = x^2 \quad , \quad y_2' = 2x \, , \quad y_2'' = 2$

$$x^2 y_2'' - 4x\, y_2' + 6y_2 = x^2(2) - 4x\,(2x) + 6x^2$$
$$= 2x^2 - 8x^2 + 6x^2 = 0$$

Both functions in (c) solve the ODE.

Check for lin. independence w/ Wronskian

$$W(y_1, y_2)(x) = \begin{vmatrix} x^3 & x^2 \\ 3x^2 & 2x \end{vmatrix} = x^3(2x) - 3x^2(x^2)$$
$$= 2x^4 - 3x^4 = -x^4 \neq 0$$

The functions in option (c) are linearly independent.

Hence option (c) works,

$\{x^3, x^2\}$ is a fundamental

solution set.

Nonhomogeneous Equations

Now we will consider the equation

$$a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

where g is not the zero function. We'll continue to assume that a_n doesn't vanish and that a_i and g are continuous.

The **associated homogeneous equation** is

$$a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = 0.$$

Write the associated homogeneous equation

(a) $x^3y'''-2x^2y''+3xy'+17y = e^{2x}$

$$x^3y''' - 2x^2y'' + 3xy' + 17y = 0$$

(b) $\dfrac{d^2y}{dx^2}+14\dfrac{dy}{dx} = \cos\left(\dfrac{\pi x}{2}\right)$

$$\frac{d^2y}{dx^2} + 14\frac{dy}{dx} = 0$$

Theorem: General Solution of Nonhomogeneous Equation

Let y_p be any solution of the nonhomogeneous equation, and let y_1, $y_2, \ldots, y_n$ be any fundamental solution set of the associated homogeneous equation.

Then the general solution of the nonhomogeneous equation is

$$y = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x) + y_p(x)$$

where c_1, $c_2, \ldots, c_n$ are arbitrary constants.

Note the form of the solution $y_c + y_p$!
(complementary plus particular)

Another Superposition Principle (for nonhomogeneous eqns.)

Let y_{p_1}, y_{p_2}, ..., y_{p_k} be k particular solutions to the nonhomogeneous linear equations

$$a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g_i(x)$$

for $i = 1, \ldots, k$. Assume the domain of definition for all k equations is a common interval I.

Then

$$y_p = y_{p_1} + y_{p_2} + \cdots + y_{p_k}$$

is a particular solution of the nonhomogeneous equation

$$a_n(x)\frac{d^ny}{dx^n} + \cdots + a_0(x)y = g_1(x) + g_2(x) + \cdots + g_k(x).$$

Example $x^2y'' - 4xy' + 6y = 36 - 14x$

(a) Verify that

$$y_{p_1} = 6 \quad \text{solves} \quad x^2y'' - 4xy' + 6y = 36.$$

$y_{p_1}' = 0$

$y_{p_1}'' = 0$

$$x^2 y_{p_1}'' - 4x y_{p_1}' + 6y_{p_1} \overset{?}{=} 36$$

$$x^2(0) - 4x(0) + 6(6) \overset{?}{=} 36$$

$$36 = 36 \quad \checkmark$$

so y_{p_1} does solve this eqn.

Example $x^2y'' - 4xy' + 6y = 36 - 14x$

(b) Verify that

$$y_{p_2} = -7x \quad \text{solves} \quad x^2y'' - 4xy' + 6y = -14x.$$

$y_{p_2}' = -7$

$y_{p_2}'' = 0$

$$x^2 y_{p_2}'' - 4x y_{p_2}' + 6y_{p_2} \stackrel{?}{=} -14x$$

$$x^2(0) - 4x(-7) + 6(-7x) \stackrel{?}{=} -14x$$

$$28x - 42x \stackrel{?}{=} -14x$$

$$-14x = -14x \quad \checkmark$$

So y_{p_2} solves the eqn.

Example $x^2y'' - 4xy' + 6y = 36 - 14x$

(c) Recall that $y_1 = x^2$ and $y_2 = x^3$ is a fundamental solution set of

$$x^2y'' - 4xy' + 6y = 0.$$

Use this along with results (a) and (b) to write the general solution of $x^2y'' - 4xy' + 6y = 36 - 14x$.

By principle of superposition $y_p = y_{p_1} + y_{p_2}$

So $y_p = 6 - 7x$

Also $y_c = c_1y_1 + c_2y_2 = c_1x^2 + c_2x^3$

The general solution for the nonhomogeneous eqn is

$$\boxed{y = c_1x^2 + c_2x^3 + 6 - 7x.}$$