

## Section 4.1 Some Theory of Linear Equations

**Definition of Wronskian** Let  $f_1, f_2, \dots, f_n$  possess at least  $n - 1$  continuous derivatives on an interval  $I$ . The **Wronskian** of this set of functions is the determinant

$$W(f_1, f_2, \dots, f_n)(x) = \begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ f'_1 & f'_2 & \cdots & f'_n \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \cdots & f_n^{(n-1)} \end{vmatrix}.$$

## Determine the Wronskian of the Functions

$$f_1(x) = xe^x, \quad f_2(x) = e^x$$

$$f_1'(x) = e^x + xe^x, \quad f_2'(x) = e^x$$

$$W(f_1, f_2)(x) = \begin{vmatrix} xe^x & e^x \\ e^x + xe^x & e^x \end{vmatrix}$$

$$= xe^x(e^x) - (e^x + xe^x) \cdot e^x$$

$$= xe^{2x} - (e^{2x} + xe^{2x})$$

$$= xe^{2x} - e^{2x} - xe^{2x} = -e^{2x}$$

$$W(xe^x, e^x)(x) = -e^{2x}$$

## An Observation

The set  $\{\sin x, \cos x\}$  is linearly independent on  $(-\infty, \infty)$  and we found that

$$W(\sin x, \cos x)(x) = -1.$$

The set  $\{x^2, 4x, x - x^2\}$  is linearly dependent on  $(-\infty, \infty)$  and we found that

$$W(x^2, 4x, x - x^2)(x) = 0.$$

## Theorem (a test for linear independence)

Let  $f_1, f_2, \dots, f_n$  be  $n - 1$  times continuously differentiable on an interval  $I$ . If there exists  $x_0$  in  $I$  such that  $W(f_1, f_2, \dots, f_n)(x_0) \neq 0$ , then the functions are **linearly independent** on  $I$ .

If  $y_1, y_2, \dots, y_n$  are  $n$  solutions of the linear homogeneous  $n^{th}$  order equation on an interval  $I$ , then the solutions are **linearly independent** on  $I$  if and only if  $W(y_1, y_2, \dots, y_n)(x) \neq 0$  for<sup>1</sup> each  $x$  in  $I$ .

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<sup>1</sup>For solutions of one linear homogeneous ODE, the Wronskian is either always zero or is never zero.

Determine if the functions are linearly dependent or independent:

$$y_1 = e^x, \quad y_2 = e^{-2x} \quad I = (-\infty, \infty)$$

$$y_1' = e^x$$

$$y_2' = -2e^{-2x}$$

We can use the Wronskian

$$W(y_1, y_2)(x) = \begin{vmatrix} e^x & e^{-2x} \\ e^x & -2e^{-2x} \end{vmatrix}$$

$$= e^x(-2e^{-2x}) - e^x(e^{-2x})$$

$$= -2e^{-x} - e^{-x}$$

$$= -3e^{-x}$$

$$W(y_1, y_2)(x) = -3e^{-x} \neq 0 \quad \text{for all real } x$$

Hence these functions are linearly independent.

\* Since  $W(y_1, y_2)(x) \neq 0$  for at least one  $x$  in  $(-\infty, \infty)$ , we can conclude lin. independence.

## Fundamental Solution Set

We're still considering this equation

$$a_n(x) \frac{d^n y}{dx^n} + a_{n-1}(x) \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x) \frac{dy}{dx} + a_0(x)y = 0$$

with the assumptions  $a_n(x) \neq 0$  and  $a_i(x)$  are continuous on  $I$ .

**Definition:** A set of functions  $y_1, y_2, \dots, y_n$  is a **fundamental solution set** of the  $n^{\text{th}}$  order homogeneous equation provided they

- (i) are solutions of the equation,
- (ii) there are  $n$  of them, and
- (iii) they are linearly independent.

**Theorem:** Under the assumed conditions, the equation has a fundamental solution set.

## General Solution of $n^{th}$ order Linear Homogeneous Equation

Let  $y_1, y_2, \dots, y_n$  be a fundamental solution set of the  $n^{th}$  order linear homogeneous equation. Then the **general solution** of the equation is

$$y(x) = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x),$$

where  $c_1, c_2, \dots, c_n$  are arbitrary constants.

## Example

Verify that  $y_1 = e^x$  and  $y_2 = e^{-x}$  form a fundamental solution set of the ODE

$$y'' - y = 0 \quad \text{on } (-\infty, \infty),$$

and determine the general solution.

There are 2 functions and a second order equation,  
so condition (ii) is satisfied.

We need to show that they both solve the D.E.

$$y_1 = e^x, \quad y_1' = e^x, \quad y_1'' = e^x$$

$$\text{so} \quad y_1'' - y_1 = e^x - e^x = 0$$

$y_1$  solves the equation.

$$y_2 = e^{-x}, \quad y_2' = -e^{-x}, \quad y_2'' = e^{-x}$$

$$y_2'' - y_2 = e^{-x} - e^{-x} = 0$$

$y_2$  solves the equation too.

Finally, we have to show that they are linearly independent. We'll use the Wronskian

$$W(y_1, y_2)(x) = \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix} = e^x(-e^{-x}) - e^x(e^{-x})$$

$$= -1 - 1 = -2$$

$W(y_1, y_2)(x) = -2 \neq 0$  for all  $x$ . Hence they  
are linearly independent.

$y_1, y_2$  form a fundamental solution  
set. The general solution is

$$y = C_1 y_1(x) + C_2 y_2(x)$$

i.e.  $y(x) = C_1 e^{x} + C_2 e^{-x}$ .

Consider  $x^2y'' - 4xy' + 6y = 0$  for  $x > 0$

2<sup>nd</sup> order equation

Determine which if any of the following sets of functions is a fundamental solution set.

- (a)  $y_1 = 2x^2, \quad y_2 = x^2$  ← not possible, lin. dependent note  
 $y_1(x) - 2y_2(x) = 0$  for all  $x > 0$
- (b)  $y_1 = x^2, \quad y_2 = x^{-2}$  ←  $y_2$  doesn't solve the DE
- (c)  $y_1 = x^3, \quad y_2 = x^2$
- (d)  $y_1 = x^2, \quad y_2 = x^3, \quad y_3 = x^{-2}$  ← not possible, too many functions

Check option (b)

$$y_2 = x^{-2}, \quad y_2' = -2x^{-3}, \quad y_2'' = 6x^{-4}$$

$$x^2 y_2'' - 4x y_2' + 6y_2 = 0$$

$$x^2(6x^{-4}) - 4x(-2x^{-3}) + 6x^{-2} = 6x^{-2} + 8x^{-2} + 6x^{-2} = 20x^{-2} \neq 0$$

$y_2$  doesn't solve the equation

option (b) can't be the answer

Option (c):  $y_1 = x^3$ ,  $y_1' = 3x^2$ ,  $y_1'' = 6x$

$$x^2 y_1'' - 4x y_1' + 6y_1 = 0$$

$$x^2(6x) - 4x(3x^2) + 6x^3 = 6x^3 - 12x^3 + 6x^3 = 0$$

$y_1$  is a solution.

$$y_2 = x^2, \quad y_2' = 2x, \quad y_2'' = 2$$

$$x^2 y_2'' - 4x y_2' + 6y_2 = ?$$

$$x^2(2) - 4x(2x) + 6x^2 = 2x^2 - 8x^2 + 6x^2 = 0$$

$y_2$  is a solution.

Check for linear Independence w/ the Wronskian

$$W(y_1, y_2)(x) = \begin{vmatrix} x^3 & x^2 \\ 3x^2 & 2x \end{vmatrix}$$

$$= x^3(2x) - 3x^2(x^2) = 2x^4 - 3x^4 = -x^4$$

$$W(y_1, y_2)(x) = -x^4 \neq 0 \text{ for all } x > 0$$

Hence  $y_1, y_2$  are linearly independent.

Option (c) is a fundamental solution set.

## Nonhomogeneous Equations

Now we will consider the equation

$$a_n(x) \frac{d^n y}{dx^n} + a_{n-1}(x) \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x) \frac{dy}{dx} + a_0(x)y = g(x)$$

where  $g$  is not the zero function. We'll continue to assume that  $a_n$  doesn't vanish and that  $a_i$  and  $g$  are continuous.

The **associated homogeneous equation** is

$$a_n(x) \frac{d^n y}{dx^n} + a_{n-1}(x) \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x) \frac{dy}{dx} + a_0(x)y = 0.$$

Write the associated homogeneous equation

(a)  $x^3y''' - 2x^2y'' + 3xy' + 17y = e^{2x}$

$$x^3y''' - 2x^2y'' + 3xy' + 17y = 0$$

(b)  $\frac{d^2y}{dx^2} + 14\frac{dy}{dx} = \cos\left(\frac{\pi x}{2}\right)$

$$\frac{d^2y}{dx^2} + 14\frac{dy}{dx} = 0$$

## Theorem: General Solution of Nonhomogeneous Equation

Let  $y_p$  be any solution of the nonhomogeneous equation, and let  $y_1, y_2, \dots, y_n$  be any fundamental solution set of the associated homogeneous equation.

Then the general solution of the nonhomogeneous equation is

$$y = \underbrace{c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x)}_{y_c} + y_p(x)$$

where  $c_1, c_2, \dots, c_n$  are arbitrary constants.

Note the form of the solution  $y_c + y_p!$   
(complementary plus particular)

## Another Superposition Principle (for nonhomogeneous eqns.)

Let  $y_{p_1}, y_{p_2}, \dots, y_{p_k}$  be  $k$  particular solutions to the nonhomogeneous linear equations

$$a_n(x) \frac{d^n y}{dx^n} + a_{n-1}(x) \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x) \frac{dy}{dx} + a_0(x)y = g_i(x)$$

for  $i = 1, \dots, k$ . Assume the domain of definition for all  $k$  equations is a common interval  $I$ .

Then

$$y_p = y_{p_1} + y_{p_2} + \cdots + y_{p_k}$$

is a particular solution of the nonhomogeneous equation

$$a_n(x) \frac{d^n y}{dx^n} + \cdots + a_0(x)y = g_1(x) + g_2(x) + \cdots + g_k(x).$$