September 17 MATH 1113 sec. 51 Fall 2018

Algebra of Rational Expressions (JIT 22, 23, 24)

A **rational expression** is a ratio (i.e. a fraction) in which the numerator and denominator are polynomial expressions. Examples include

$$\frac{x^3+4x^2}{x^2-16} \quad \text{and} \quad \frac{2a}{a+3}.$$

We will study *rational functions* in the next section. So here we refresh the basic algebra (addition, subtraction, multiplication, division) and simplification involving rational expressions.

It draws on our knowledge of rational numbers and polynomials.

Example

Evaluate the sum and simplify if possible.

$$\frac{x}{x^2-1} - \frac{2}{x^2+2x-3}$$

We need a common denominator.

We'll use a least common denominator (LCD).

We'll factor our denominators

$x^2-1 = (x-1)(x+1)$

$x^2+2x-3 = (x-1)(x+3)$

we'll need factors

$x-1,\ x+1,\ x+3$

our LCD is

$(x-1)(x+1)(x+3)$

$$\frac{x}{(x-1)(x+1)} - \frac{2}{(x-1)(x+3)} =$$

$$\frac{x}{(x-1)(x+1)}\left(\frac{x+3}{x+3}\right) - \frac{2}{(x-1)(x+3)}\left(\frac{x+1}{x+1}\right)$$

$$\frac{x(x+3)}{(x-1)(x+1)(x+3)} - \frac{2(x+1)}{(x-1)(x+1)(x+3)} =$$

$$\frac{x^2+3x - (2x+2)}{(x-1)(x+1)(x+3)} =$$

$$\frac{x^2+3x-2x-2}{(x-1)(x+1)(x+3)} = \frac{x^2+x-2}{(x-1)(x+1)(x+3)}$$

$$= \frac{(x+2)\cancel{(x-1)}}{\cancel{(x-1)}(x+1)(x+3)}$$

for $x-1 \neq 0$ cancel $x-1$

$$= \frac{x+2}{(x+1)(x+3)}$$

Question

Suppose we wish to evaluate the sum

$$\frac{-8}{x^2-4}+\frac{2}{x-2}.$$

The **least common denominator** we can use for this operation is

(a) $(x^2-4)(x-2)$

$x^2-4 = (x-2)(x+2)$

(b) (x^2+x-6)

$x-2 = x-2$

(c) $(x-2)(x+2)$

(d) $(x-2)^2(x+2)$

Question

$$\frac{-8}{x^2-4}+\frac{2}{x-2} = \frac{-8}{(x-2)(x+2)} + \frac{2}{x-2}\cdot\frac{x+2}{x+2}$$

(a) $\dfrac{-6}{x^2-4}$

(b) $\dfrac{2x+10}{x^2-4}$

(c) $\dfrac{-6}{x^2+x-6}$

(d) $\dfrac{2}{x+2}$

$$= \frac{-8+(2x+4)}{(x-2)(x+2)}$$

$$= \frac{-8+2x+4}{(x-2)(x+2)} = \frac{2x-4}{(x-2)(x+2)}$$

$$= \frac{2(x-2)}{(x-2)(x+2)} = \frac{2}{x+2}$$

Simplifying Complex Rational Expressions

A **complex**[1] rational expression is one in which the numerator or denominator (or both) contain a rational expression—i.e. fraction within a fraction. Examples include

$$\frac{\frac{1}{x}+1}{1-\frac{1}{x}} \quad \text{and} \quad \frac{\frac{w}{v}+\frac{v}{w}}{wv}$$

We wish to rewrite such as thing as a rational expression that is no longer complex.

[1] also called *compound* rational expressions

Example: Simplify the complex rational expression.

We'll use two different approaches. First, we'll simplify numerator and denominator before performing the division.

$$\frac{\frac{1}{x}-\frac{1}{y}}{\frac{1}{x^2}-\frac{1}{y^2}}$$

work with $\frac{1}{x}-\frac{1}{y}$

$$\frac{1}{x}\frac{y}{y}-\frac{1}{y}\frac{x}{x}=\frac{y}{xy}-\frac{x}{xy}=\frac{y-x}{xy}$$

work with $\frac{1}{x^2}-\frac{1}{y^2}$

$$\frac{1}{x^2}\frac{y^2}{y^2}-\frac{1}{y^2}\frac{x^2}{x^2}=\frac{y^2}{x^2y^2}-\frac{x^2}{y^2x^2}$$

$$=\frac{y^2-x^2}{x^2y^2}$$

$$\frac{\dfrac{1}{x} - \dfrac{1}{y}}{\dfrac{1}{x^2} - \dfrac{1}{y^2}} = \frac{\dfrac{y-x}{xy}}{\dfrac{y^2-x^2}{x^2y^2}} = \frac{y-x}{\cancel{x}\,\cancel{y}} \cdot \frac{\cancel{x^2}\,\cancel{y^2}}{y^2-x^2}$$

$$= \frac{(y-x)\,xy}{y^2-x^2}$$

$$= \frac{\cancel{(y-x)}\,xy}{\cancel{(y-x)}(y+x)} = \frac{xy}{y+x}$$

Example: Simplify the complex rational expression.

Now we do this again by **clearing the fractions**.

$$\frac{\frac{1}{x}-\frac{1}{y}}{\frac{1}{x^2}-\frac{1}{y^2}}$$

Find the LCD of all rational expressions within this one.

This is x^2y^2. multiply the whole expression by $\frac{LCD}{LCD}$

$$\frac{\frac{1}{x}-\frac{1}{y}}{\frac{1}{x^2}-\frac{1}{y^2}}\left(\frac{x^2y^2}{x^2y^2}\right)$$

$$= \frac{\frac{1}{x}(x^2y^2) - \frac{1}{y}(x^2y^2)}{\frac{1}{x^2}(x^2y^2) - \frac{1}{y^2}(x^2y^2)} = \frac{xy^2 - x^2y}{y^2 - x^2}$$

$$= \frac{xy(y-x)}{y^2 - x^2}$$

$$= \frac{xy(y-x)}{(y-x)(y+x)} = \frac{xy}{y+x}$$