Sept 17 Math 2306 sec. 53 Fall 2018

Section 6: Linear Equations Theory and Terminology

Definition of Wronskian Let $f_1, f_2, \ldots, f_n$ posses at least $n-1$ continuous derivatives on an interval I. The **Wronskian** of this set of functions is the determinant

$$W(f_1, f_2, \ldots, f_n)(x) = \begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ \vdots & \vdots & \vdots & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \cdots & f_n^{(n-1)} \end{vmatrix}.$$

(Note that, in general, this Wronskian is a function of the independent variable x.)

Determinants

If A is a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then its determinant

$$\det(A) = ad - bc.$$

If A is a 3×3 matrix $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, then its determinant

$$\det(A) = a_{11}\det\begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} - a_{12}\det\begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix} + a_{13}\det\begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$$

Determine the Wronskian of the Functions

$$f_1(x) = \sin x, \quad f_2(x) = \cos x$$

We computed the Wronskian

$$W(f_1, f_2)(x) = \begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix} = -1.$$

Determine the Wronskian of the Functions

$$f_1(x) = x^2, \quad f_2(x) = 4x, \quad f_3(x) = x - x^2$$

3 functions $\rightarrow$ 3x3 matrix

$$W(f_1, f_2, f_3)(x) = \begin{vmatrix} x^2 & 4x & x - x^2 \\ 2x & 4 & 1-2x \\ 2 & 0 & -2 \end{vmatrix}$$

$$= x^2 \begin{vmatrix} 4 & 1-2x \\ 0 & -2 \end{vmatrix} - 4x \begin{vmatrix} 2x & 1-2x \\ 2 & -2 \end{vmatrix} + (x-x^2) \begin{vmatrix} 2x & 4 \\ 2 & 0 \end{vmatrix}$$

$$= x^2\left(4\cdot(-2) - 0\cdot(1-2x)\right) - 4x\left(2x(-2) - 2(1-2x)\right) + (x-x^2)\left(2x\cdot 0 - 2\cdot 4\right)$$

$$= x^2(-8) - 4x(-4x - 2 + 4x) + (x-x^2)(-8)$$

$$= -8x^2 + 8x - 8x + 8x^2$$

$$= 0$$

$$W(f_1, f_2, f_3)(x) = 0$$

Theorem (a test for linear independence)

Let $f_1, f_2, \ldots, f_n$ be $n-1$ times continuously differentiable on an interval I. If there exists x_0 in I such that $W(f_1, f_2, \ldots, f_n)(x_0) \neq 0$, then the functions are **linearly independent** on I.

functions are lin. dependent if $W(x) = 0$ for all x in I

If $y_1, y_2, \ldots, y_n$ are n solutions of the linear homogeneous n^{th} order equation on an interval I, then the solutions are **linearly independent** on I if and only if $W(y_1, y_2, \ldots, y_n)(x) \neq 0$ for[1] each x in I.

[1] For solutions of one linear homogeneous ODE, the Wronskian is either always zero or is never zero.

Determine if the functions are linearly dependent or independent:

$y_1 = e^x, \quad y_2 = e^{-2x} \quad I = (-\infty, \infty)$

We can use the Wronskian

$$W(y_1, y_2)(x) = \begin{vmatrix} e^x & e^{-2x} \\ e^x & -2e^{-2x} \end{vmatrix} = e^x(-2e^{-2x}) - e^x(e^{-2x})$$

$$= -2e^{-x} - e^{-x} = -3e^{-x}$$

$W(y_1, y_2)(x) = -3e^{-x}$ Non zero

The functions are linearly independent.

Fundamental Solution Set

We're still considering this equation

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = 0$$

with the assumptions $a_n(x) \neq 0$ and $a_i(x)$ are continuous on I.

Definition: A set of functions $y_1, y_2, \ldots, y_n$ is a **fundamental solution set** of the n^{th} order homogeneous equation provided they

(i) are solutions of the equation,

(ii) there are n of them, and

(iii) they are linearly independent.

Theorem: Under the assumed conditions, the equation has a fundamental solution set.

General Solution of n^{th} order Linear Homogeneous Equation

Let $y_1, y_2, \ldots, y_n$ be a fundamental solution set of the n^{th} order linear homogeneous equation. Then the **general solution** of the equation is

$$y(x) = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x),$$

where $c_1, c_2, \ldots, c_n$ are arbitrary constants.

Example

Verify that $y_1 = x^2$ and $y_2 = x^3$ form a fundamental solution set of the ODE

$$x^2y'' - 4xy' + 6y = 0 \quad \text{on} \quad (0, \infty),$$

and determine the general solution.

2 functions and 2^{nd} order equation (right # of functions)

Verify they are solutions:

$y_1 = x^2$

$y_1' = 2x$ substitute

$y_1'' = 2$

$x^2 y_1'' - 4x y_1' + 6y_1 =$

$x^2(2) - 4x(2x) + 6(x^2) =$

$2x^2 - 8x^2 + 6x^2 =$

$0 = 0$

y_1 solves the ODE

$y_2 = x^3$

$y_2' = 3x^2$ Substitute

$y_2'' = 6x$

$x^2 y_2'' - 4x\, y_2' + 6y_2 =$

$x^2(6x) - 4x\,(3x^2) + 6x^3 = 0$

$6x^3 - 12x^3 + 6x^3 =$

$0 = 0$

y_2 solves the ODE

Finally, we'll show they are linearly independent.

Using the Wronskian:

$$W(y_1, y_2)(x) = \begin{vmatrix} x^2 & x^3 \\ 2x & 3x^2 \end{vmatrix}$$

$$= x^2(3x^2) - 2x(x^3)$$

$$= 3x^4 - 2x^4 \quad = x^4$$

$W(y_1, y_2)(x) = x^4 \neq 0$ they are independent!

The general solution is

$$y = c_1 y_1 + c_2 y_2$$

$$y = c_1 x^2 + c_2 x^3$$