September 18 Math 2306 sec 51 Fall 2015

Section 4.1 Some Theory of Linear Equations

Solve the IVP

$$x^2y'' - 4xy' + 6y = 36 - 14x, \quad y(1) = 0, \quad y'(1) = -5$$

From last time, we found the general solution to be
$y = c_1x^2 + c_2x^3 + 6 - 7x$

$y' = 2c_1 x + 3c_2 x^2 - 7$

$y(1) = c_1(1)^2 + c_2(1)^3 + 6 - 7(1) = 0$

$c_1 + c_2 - 1 = 0 \implies c_1 + c_2 = 1$

$y'(1) = 2c_1(1) + 3c_2(1)^2 - 7 = -5$

$2c_1 + 3c_2 - 7 = -5 \implies 2c_1 + 3c_2 = 2$

$$C_1 + C_2 = 1 \quad\Rightarrow\quad -2C_1 - 2C_2 = -2$$

$$2C_1 + 3C_2 = 2 \qquad\qquad 2C_1 + 3C_2 = 2 \quad \text{add}$$

$$C_2 = 0$$

$$C_1 + 0 = 1 \Rightarrow C_1 = 1$$

So the solution to the IVP is

$$\boxed{y = x^2 + 6 - 7x}$$

Section 4.2: Reduction of Order

We'll focus on second order, linear, homogeneous equations. Recall that such an equation has the form

$$a_2(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y = 0.$$

Let us assume that $a_2(x) \neq 0$ on the interval of interest. We will write our equation in **standard form**

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0$$

where $P = a_1/a_2$ and $Q = a_0/a_2$.

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0$$

Recall that every fundmantal solution set will consist of two linearly independent solutions y_1 and y_2, and the general solution will have the form

$$y = c_1 y_1(x) + c_2 y_2(x).$$

Suppose we happen to know one solution $y_1(x)$. **Reduction of order** is a method for finding a second linearly independent solution $y_2(x)$ that starts with the assumption that

$$y_2(x) = u(x) y_1(x)$$

for some function $u(x)$. The method involves finding the function u.

Example

Verify that $y_1 = e^{-x}$ is a solution of $y'' - y = 0$. Then find a second solution y_2 of the form

$$y_2(x) = u(x)y_1(x) = e^{-x}u(x).$$

Confirm that the pair y_1, y_2 is linearly independent.

Verify y_1 solves the D.E. : $y_1 = e^{-x}$, $y_1' = -e^{-x}$, $y_1'' = e^{-x}$

$y_1'' - y_1 = e^{-x} - e^{-x} = 0 \Rightarrow y_1$ solves the equation.

Set $y_2(x) = e^{-x}u(x)$. To sub it into the DE, we need $y_2''(x)$

$$y_2(x) = e^{-x}u(x)$$

$$y_2'(x) = e^{-x}u'(x) - e^{-x}u(x)$$

$$y_2''(x) = e^{-x}u''(x) - e^{-x}u'(x) - e^{-x}u'(x) + e^{-x}u(x)$$

$$= e^{-x}u''(x) - 2e^{-x}u'(x) + e^{-x}u(x)$$

Need $\quad y_2'' - y_2 = 0 \Rightarrow$

$$e^{-x}u'' - 2e^{-x}u' + \cancel{e^{-x}u} - \cancel{e^{-x}u} = 0$$

$$e^{-x}u'' - 2e^{-x}u' = 0 \quad \Rightarrow$$

$$\Rightarrow \quad u'' - 2u' = 0 \qquad \text{(Divide out } e^{-x}\text{)}$$

This is 1^{st} order linear in u'. Let $w = u'$.

Then $w' = u''$ so

$$w' - 2w = 0 \implies \frac{dw}{dx} = 2w$$

$$\frac{1}{w}\frac{dw}{dx} = 2 \implies \frac{1}{w}\frac{dw}{dx}\,dx = 2\,dx$$

$$\int \frac{1}{w}\,dw = \int 2\,dx$$

$$\ln w = 2x \implies w = e^{2x}$$

Since $w = u'$, $u = \int w\,dx$

So $\quad u = \int e^{2x}\,dx = \frac{1}{2} e^{2x}$

From $y_2(x) = u(x)\,y_1(x) = e^{-x} u(x)$, we

find $\quad y_2(x) = e^{-x}\left(\frac{1}{2} e^{2x}\right) = \frac{1}{2} e^{x}$

Now we can verify that y_1 and y_2 are

linearly independent.

$$W(y_1, y_2)(x) = \begin{vmatrix} e^{-x} & \frac{1}{2}e^{x} \\ -e^{-x} & \frac{1}{2}e^{x} \end{vmatrix}$$

$$= e^{-x}\left(\tfrac{1}{2}e^{x}\right) - \left(-e^{-x}\right)\left(\tfrac{1}{2}e^{x}\right) = \tfrac{1}{2} + \tfrac{1}{2} = 1$$

Since $W(y_1, y_2)(x) = 1 \neq 0$, y_1 and y_2 are linearly independent.

The general solution is $y = c_1 y_1 + c_2 y_2$. We can absorb the $\frac{1}{2}$ into c_2 and take $y_2 = e^{x}$.

Generalization

Consider the equation **in standard form** with one known solution. Determine a second linearly independent solution.

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0, \quad y_1(x) - -\text{is known.}$$

Set $y_2(x) = u(x)\,y_1(x)$

$y_2' = u' y_1 + u y_1'$

$y_2'' = u'' y_1 + u' y_1' + u' y_1' + u y_1''$

$y_2'' = u'' y_1 + 2u' y_1' + u y_1''$

we'll continue on Monday.