September 18 Math 2306 sec 54 Fall 2015

Section 4.1 Some Theory of Linear Equations

General Solution of Nonhomogeneous Equation
Let y_p be any solution of the nonhomogeneous equation, and let y_1, $y_2, \ldots, y_n$ be any fundamental solution set of the associated homogeneous equation.

Then the general solution of the nonhomogeneous equation is

$$y = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x) + y_p(x)$$

where c_1, $c_2, \ldots, c_n$ are arbitrary constants.

Superposition Principle for Nonhomogeneous Equations

Let y_{p_1}, y_{p_2}, ..., y_{p_k} be k particular solutions to the nonhomogeneous linear equations

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g_i(x)$$

for $i = 1, \ldots, k$. Assume the domain of definition for all k equations is a common interval I.

Then

$$y_p = y_{p_1} + y_{p_2} + \cdots + y_{p_k}$$

is a particular solution of the nonhomogeneous equation

$$a_n(x)\frac{d^n y}{dx^n} + \cdots + a_0(x)y = g_1(x) + g_2(x) + \cdots + g_k(x).$$

Example $x^2y'' - 4xy' + 6y = 36 - 14x$

(a) Verify that

$$y_{p_1} = 6 \quad \text{solves} \quad x^2y'' - 4xy' + 6y = 36.$$

$y_{p_1}' = 0$

$y_{p_1}'' = 0$

$$x^2 y_p'' - 4x y_p' + 6 y_p \stackrel{?}{=} 36$$

$$x^2(0) - 4x(0) + 6(6) \stackrel{?}{=} 36$$

$$36 = 36 \quad \checkmark$$

So $y_{p_1} = 6$ solves the nonhomogeneous equation.

Example $x^2y'' - 4xy' + 6y = 36 - 14x$

(b) Verify that

$$y_{p_2} = -7x \quad \text{solves} \quad x^2y'' - 4xy' + 6y = -14x.$$

$y_{p_2}' = -7$

$y_{p_2}'' = 0$

$$x^2 y_{p_2}'' - 4x y_{p_2}' + 6y_{p_2} \stackrel{?}{=} -14x$$

$$x^2(0) - 4x(-7) + 6(-7x) \stackrel{?}{=} -14x$$

$$28x - 42x \stackrel{?}{=} -14x$$

$$-14x = -14x \quad \checkmark$$

So y_{p_2} solves this nonhomogeneous eqn.

Example $x^2y'' - 4xy' + 6y = 36 - 14x$

(c) Recall that $y_1 = x^2$ and $y_2 = x^3$ is a fundamental solution set of

$$x^2y'' - 4xy' + 6y = 0.$$

Use this along with results (a) and (b) to write the general solution of $x^2y'' - 4xy' + 6y = 36 - 14x$.

We can take $y_p = y_{p_1} + y_{p_2} = 6 - 7x$

The general solution is

$$y = c_1 x^2 + c_2 x^3 + 6 - 7x$$

Solve the IVP

$$x^2y'' - 4xy' + 6y = 36 - 14x, \quad y(1) = 0, \quad y'(1) = -5$$

Our general solution is

$$y = c_1 x^2 + c_2 x^3 + 6 - 7x$$

$$y' = 2c_1 x + 3c_2 x^2 - 7$$

$$y(1) = c_1(1)^2 + c_2(1)^3 + 6 - 7(1) = 0$$

$$c_1 + c_2 - 1 = 0 \quad \Rightarrow \quad c_1 + c_2 = 1$$

$$y'(1) = 2c_1(1) + 3c_2(1)^2 - 7 = -5$$

$$2c_1 + 3c_2 = 2$$

$$c_1 + c_2 = 1 \quad \Rightarrow \quad -2c_1 - 2c_2 = -2$$

$$2c_1 + 3c_2 = 2 \qquad\qquad 2c_1 + 3c_2 = 2 \quad \text{add}$$

$$c_1 + 0 = 1 \Rightarrow c_1 = 1 \qquad\qquad c_2 = 0$$

The solution to the IVP is

$$y = x^2 + 6 - 7x.$$

Section 4.2: Reduction of Order

We'll focus on second order, linear, homogeneous equations. Recall that such an equation has the form

$$a_2(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y = 0.$$

Let us assume that $a_2(x) \neq 0$ on the interval of interest. We will write our equation in **standard form**

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0$$

where $P = a_1/a_2$ and $Q = a_0/a_2$.

$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0$

Recall that every fundmantal solution set will consist of two linearly independent solutions y_1 and y_2, and the general solution will have the form

$$y = c_1 y_1(x) + c_2 y_2(x).$$

Suppose we happen to know one solution $y_1(x)$. **Reduction of order** is a method for finding a second linearly independent solution $y_2(x)$ that starts with the assumption that

$$y_2(x) = u(x)y_1(x)$$

for some function $u(x)$. The method involves finding the function u.

Example

Verify that $y_1 = e^{-x}$ is a solution of $y'' - y = 0$. Then find a second solution y_2 of the form

$$y_2(x) = u(x)y_1(x) = e^{-x}u(x).$$

Confirm that the pair y_1, y_2 is linearly independent.

Verify that $y_1 = e^{-x}$ solves the DE:

$y_1 = e^{-x}$, $y_1' = -e^{-x}$, $y_1'' = e^{-x}$

$y_1'' - y_1 \overset{?}{=} 0 \qquad e^{-x} - e^{-x} = 0 \qquad 0 = 0$ ✓

So y_1 does solve the D.E.

Set $y_2 = e^{-x}u(x)$ and sub into the DE.

$$y_2 = e^{-x} u$$

$$y_2' = e^{-x} u' - e^{-x} u$$

$$y_2'' = e^{-x} u'' - e^{-x} u' - e^{-x} u' + e^{-x} u$$

$$= e^{-x} u'' - 2e^{-x} u' + e^{-x} u$$

$$y_2'' - y_2 = 0$$

$$e^{-x} u'' - 2e^{-x} u' + e^{-x} \cancel{u} - e^{-x} \cancel{u} = 0$$

$$e^{-x} u'' - 2e^{-x} u' = 0 \leftarrow$$ u must solve this eqn.

This equation is 1^{st} order in u'. Let $w = u'$.

Then $w' = u''$. The DE becomes

$$e^{-x} w' - 2e^{-x} w = 0$$ divide by e^{-x}

$$w' - 2w = 0 \implies \frac{dw}{dx} = 2w$$

$$\frac{1}{w}\frac{dw}{dx}\,dx = 2\,dx$$

$$\int \frac{1}{w}\,dw = \int 2\,dx$$

$$\ln w = 2x \implies w = e^{2x}$$

Since $u' = w$, $u = \int w\,dx = \int e^{2x}\,dx = \frac{1}{2}e^{2x}$

$$y_2(x) = e^{-x}u(x) = e^{-x}\left(\frac{1}{2}e^{2x}\right) = \frac{1}{2}e^{x}$$

So $y_1 = e^{-x}$ and $y_2 = \frac{1}{2}e^{x}$

Lets verify they are linearly independent.

$$W(y_1, y_2)(x) = \begin{vmatrix} e^{-x} & \frac{1}{2}e^{x} \\ -e^{-x} & \frac{1}{2}e^{x} \end{vmatrix} = \frac{1}{2} + \frac{1}{2} = 1 \neq 0$$

Hence they are lin. independent.