# September 19 MATH 1113 sec. 51 Fall 2018

## Section 4.5: Rational Functions

**Definition:** The function $f(x)$ is a *rational function* if $f$ has the form

$$f(x) = \frac{p(x)}{q(x)}$$

where $p(x)$ and $q(x)$ are both polynomials (and $q(x)$ is not the zero function).

**Remember that division by zero is not defined!**

**Domain of a Rational Function:** The domain of $f$ is

$$\{x \in \mathbb{R} \mid q(x) \neq 0\}.$$

**Notation:** The symbol "$\in$" means *in* as in *is an element of*.

# Examples

Some examples of rational functions include:

$$f(x) = \frac{3x^2+4x-1}{x^3-x}, \quad g(x) = \frac{1}{x+1}, \quad h(x) = \frac{1}{4x+3} + 2$$

f and g are clearly ratios of polynomials

Note

$$h(x) = \frac{1}{4x+3} + 2 = \frac{1}{4x+3} + 2\left(\frac{4x+3}{4x+3}\right)$$

$$= \frac{1}{4x+3} + \frac{8x+6}{4x+3} = \frac{8x+7}{4x+3}$$

# Question

Which results in a true statement.

The function $F(x) = \dfrac{\sqrt{x}}{x^2 + 3x}$

(a) IS a rational function because it is a fraction.

(b) IS NOT a rational function because $\sqrt{x}$ is not a polynomial.

(c) IS a rational function because $\sqrt{x} = x^{1/2}$.

(d) IS NOT a rational function because its name is a capital letter $F$ instead of lowercase $f$.

# Example

Determine the domain of the rational function

$$f(x) = \frac{3x^2 + 4x - 1}{x^3 - x}$$

Here $g(x) = x^3 - x$ and our domain will be all reals such that $g(x) \neq 0$.

Find the zeros of $g$:

$g(x) = 0 \implies x^3 - x = 0$

factor $\quad x(x^2 - 1) = 0$

$x(x-1)(x+1) = 0$

This holds if $x=0$, $x-1=0$, or $x+1=0$

$x=0$, $x=1$ or $x=-1$

The domain of $f$ is

$$(-\infty,-1)\cup(-1,0)\cup(0,1)\cup(1,\infty).$$

# Graphs of Rational Functions

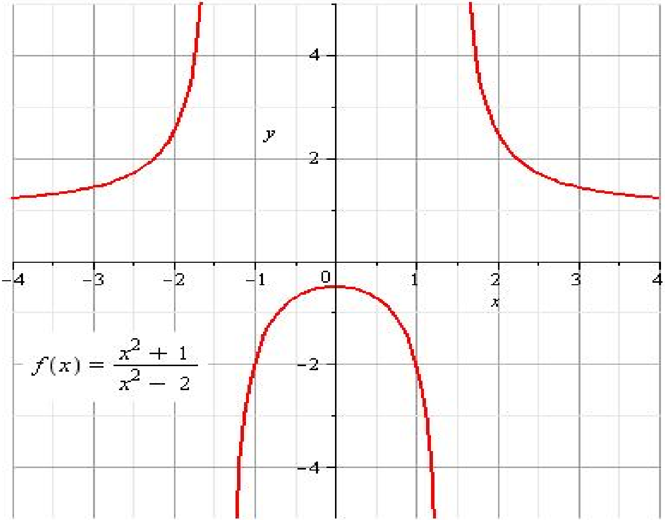



Figure: $f(x) = \frac{x^2+1}{x^2-2}$

# Graphs of Rational Functions

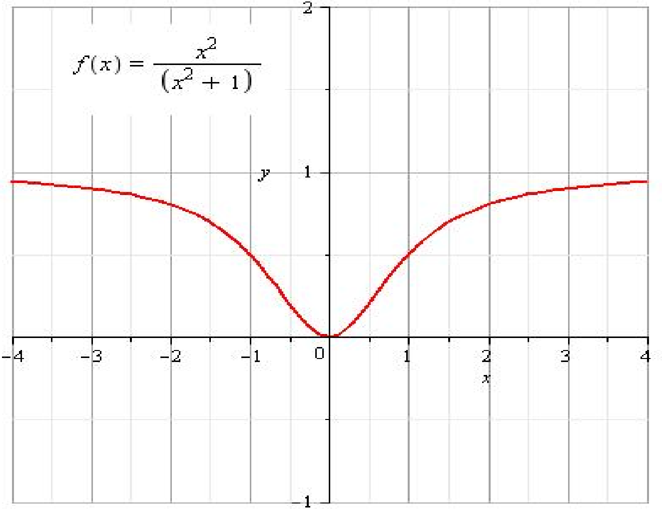



Figure: $f(x) = \frac{x^2}{x^2+1}$

# Graphs of Rational Functions

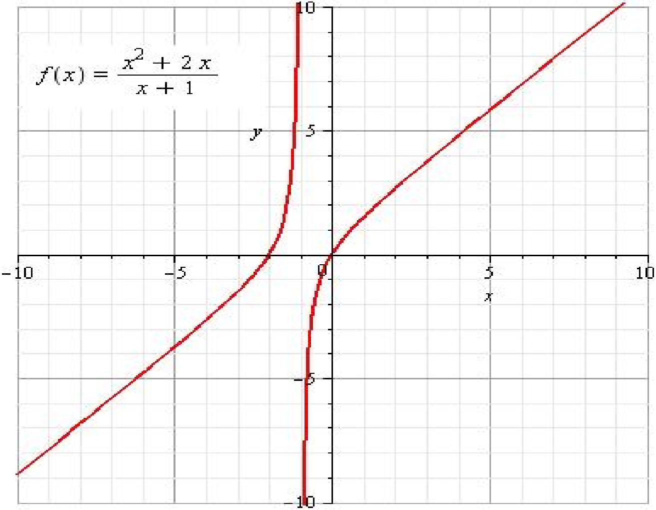



Figure: $f(x) = \frac{x^2+2x}{x+1}$

# Graphing Rational Functions: Asymptotes

Unlike a polynomial, a rational function's domain need not be all real numbers, and its graph may have asymptotes.

**Question:** How does a rational function behave near a value of $x = a$ for which $q(a) = 0$?

It turns out that there are two possibilities:

- The function values **blow up** –i.e go to $\infty$ or $-\infty$ ($p(a) \neq 0$), or
- the graph has a **hole** in it. ($p(a) = 0$).

# Vertical Asymptote

Consider the simple rational function

$$f(x) = \frac{1}{x}.$$

Domain $\{x \mid x \neq 0\}$

Determine the behavior of $f(x)$ for $x \approx 0$.

| $x$ | $f(x)$ | $x$ | $f(x)$ |
|---|---|---|---|
| 0.1 | 10 | $-0.1$ | -10 |
| 0.01 | 100 | $-0.01$ | -100 |
| 0.001 | 1000 | $-0.001$ | -1000 |
| 0.0001 | 10,000 | $-0.0001$ | -10,000 |

# Vertical Asymptote

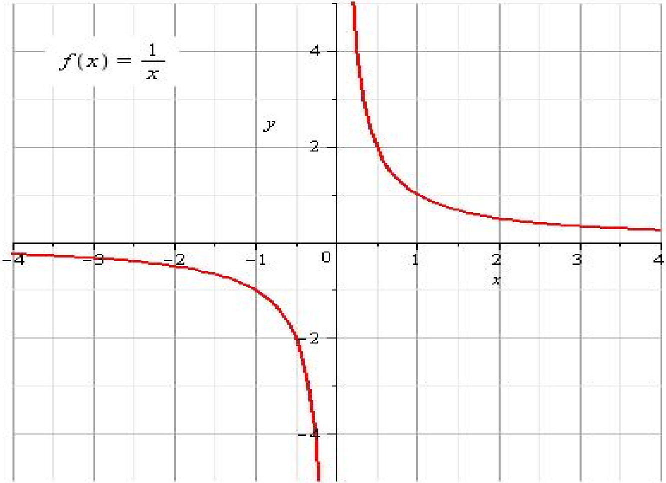



Figure: For $x \approx 0$, the graph seems to hug the vertical line $x = 0$.

# Vertical Asymptote

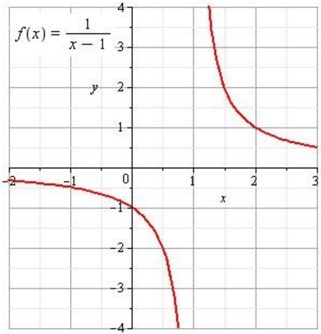
**Definition:** The vertical line $x = a$ is a vertical asymptote of the function $f$ if $f(x)$ increases without bound as $x$ approaches the number $a$.

**Remark 1:** A rational function **MAY** have a vertical asymptote at a value $a$ that is not in its domain.

**Remark 2:** The graph of a rational function **CANNOT** cross a vertical asymptote.

# Some Useful Notation

- $x \to a$ reads as "$x$ approaches $a$",
- $x \to a^-$ reads as "$x$ approaches $a$ from the left",
- $x \to a^+$ reads as "$x$ approaches $a$ from the right",
- $f(x) \to \infty$ reads as "$f$ of $x$ approaches *infinity*", and
- $f(x) \to -\infty$ reads as "$f$ of $x$ approaches negative *infinity*."




This graph shows

$f(x) \to \infty$ as $x \to 1^+$

and

$f(x) \to -\infty$ as $x \to 1^-$

# Lowest Terms

**Definition:** The rational function

$$f(x) = \frac{p(x)}{q(x)}$$

is in **lowest terms** if the polynomials $p$ and $q$ have no common factors.

Example:

$$f(x) = \frac{(x+2)(x-3)}{x^2(x+5)} \quad \text{is in lowest terms.}$$

$$g(x) = \frac{(x+2)(x-3)}{x^2(x+2)} \quad \text{is NOT in lowest terms.}$$

In lowest terms $\quad g(x) = \frac{x-3}{x^2}$ for $x \neq -2$

# Question

Consider the two rational functions $F$ and $G$.

$$F(x) = \frac{x^2-1}{x^2+x} = \frac{(x-1)(x+1)}{x(x+1)} \qquad G(x) = \frac{x^2-1}{x^2-4} = \frac{(x-1)(x+1)}{(x-2)(x+2)}$$

Which of the following is true?

(a) $F$ and $G$ are in lowest terms.

(b) Neither $F$ nor $G$ is in lowest terms.

(c) $F$ is in lowest terms, but $G$ is not in lowest terms.

(d) $F$ is not in lowest terms, but $G$ is in lowest terms.

# Finding Vertical Asymptotes

**Theorem:** Let $f(x) = \frac{p(x)}{q(x)}$ be a rational function **in lowest terms.** If $a$ is any number such that $q(a) = 0$, then

the line $x = a$ is a vertical asymptote to the graph of $f$.

**Remark:** To find vertical asymptote(s): (1) get $f$ into lowest terms, then (2) determine all values that make the denominator zero.

**Note that a vertical asymptote is a line—i.e. $x = -2$ *could* be a vertical asymptote, but $-2$ CANNOT.**

# Example: $f(x) = \frac{x^2+3x}{x^3-x}$

Determine the domain of the rational function $f$.

$$f(x) = \frac{x^2+3x}{x^3-x} = \frac{x(x+3)}{x(x-1)(x+1)}$$

The denominator is zero if $x=0$, $x=1$,

or $x=-1$.

The domain is $\{x \mid x\neq 0,\ x\neq 1, \text{ and } x\neq -1\}$

## Example: $f(x) = \frac{x^2+3x}{x^3-x}$

Express $f$ in lowest terms.

$$f(x) = \frac{x(x+3)}{x(x-1)(x+1)}$$

not in lowest terms due to common factor $x$

$$f(x) = \frac{x+3}{(x-1)(x+1)} \quad \text{for } x \neq 0$$

## Example: $f(x) = \frac{x^2+3x}{x^3-x}$

Find the equations of all vertical asymptotes to $f$.

$$f(x) = \frac{x+3}{(x-1)(x+1)} \quad \text{for} \quad x \neq 0$$

There are two vertical asymptotes

$x=1$ and $x=-1$.

* Note if $g(x) = \frac{x+3}{(x-1)(x+1)}$ then $g(0) = \frac{0+3}{(0-1)(0+1)} = -3$

So has a hole @ $(0,-3)$.

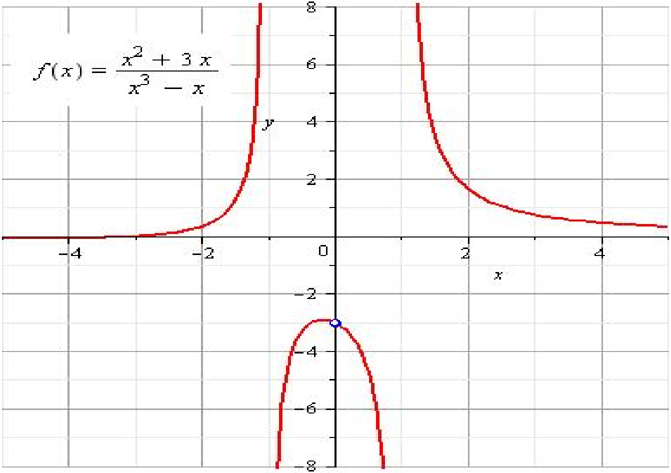



Figure: There is a hole in the graph at the point $(0, -3)$.

# Question

Determine the domain of $g$, and find all vertical asymptotes to $g$.

$$g(x) = \frac{x^2 - x - 2}{x^2 - 4} = \frac{(x-2)(x+1)}{(x-2)(x+2)}$$

(a) The domain is $(-\infty, -2) \cup (-2, \infty)$, and there is one V. asymptote $x = -2$.

(b) The domain is $\{x \in \mathbb{R} \,|\, x \neq \pm 2\}$ , and there is one V. asymptote $x = -2$.

(c) The domain is $\{x \in \mathbb{R} \,|\, x \neq \pm 2\}$ , and there are two V. asymptotes $x = -2$ and $x = 2$.

(d) The domain is $(-\infty, -2) \cup (-2, -1) \cup (-1, \infty)$, and there is one V. asymptote $x = -2$.

# Horizontal Asymptotes

Vertical asymptotes correspond to unbounded $f(x)$ values. What about when the independent variable $x$ becomes unbounded?

**Definition:** The horizontal line $y = b$ is a horizontal asymptote to the function $f$ if the graph of $f(x)$ hugs the line $y = b$ as $x$ becomes unbounded. Symbolically

$$f(x) \to b \quad \text{as} \quad x \to \infty \quad \text{or as} \quad x \to -\infty.$$