September 19 MATH 1113 sec. 52 Fall 2018

Section 4.5: Rational Functions

Definition: The function $f(x)$ is a *rational function* if f has the form

$$f(x) = \frac{p(x)}{q(x)}$$

where $p(x)$ and $q(x)$ are both polynomials (and $q(x)$ is not the zero function).

Remember that division by zero is not defined!

Domain of a Rational Function: The domain of f is

$$\{x \in \mathbb{R} \mid q(x) \neq 0\}.$$

Notation: The symbol "$\in$" means *in* as in *is an element of*.

Examples

Some examples of rational functions include:

$$f(x) = \frac{3x^2+4x-1}{x^3-x}, \quad g(x) = \frac{1}{x+1}, \quad h(x) = \frac{1}{4x+3} + 2$$

f and g are clearly ratios of polynomials.

Note that

$$h(x) = \frac{1}{4x+3} + 2 = \frac{1}{4x+3} + 2\left(\frac{4x+3}{4x+3}\right)$$

$$= \frac{1}{4x+3} + \frac{8x+6}{4x+3} = \frac{8x+7}{4x+3}$$

$h(x) = \frac{8x+7}{4x+3}$ is also a rational function.

Question

Which results in a true statement.

The function $F(x) = \dfrac{\sqrt{x}}{x^2 + 3x}$

(a) IS a rational function because it is a fraction.

(b) IS NOT a rational function because $\sqrt{x}$ is not a polynomial.

(c) IS a rational function because $\sqrt{x} = x^{1/2}$.

(d) IS NOT a rational function because its name is a capital letter F instead of lowercase f.

Example

Determine the domain of the rational function

$$f(x) = \frac{3x^2 + 4x - 1}{x^3 - x}$$

Here $g(x) = x^3 - x$. Our domain is all reals except the zeros of g.

Let's find the zeros of g.

$g(x) = 0 \implies x^3 - x = 0$

$x(x^2 - 1) = 0$

$x(x-1)(x+1) = 0$

$g(x)=0$ if $x=0$, $x-1=0$ or $x+1=0$

We have three zeros, 0, 1, and -1.

The domain is

$$(-\infty,-1)\cup(-1,0)\cup(0,1)\cup(1,\infty).$$

In set builder notation

$$\{x \mid x\neq 0,\ x\neq 1, \text{ and } x\neq -1\}$$

Graphs of Rational Functions

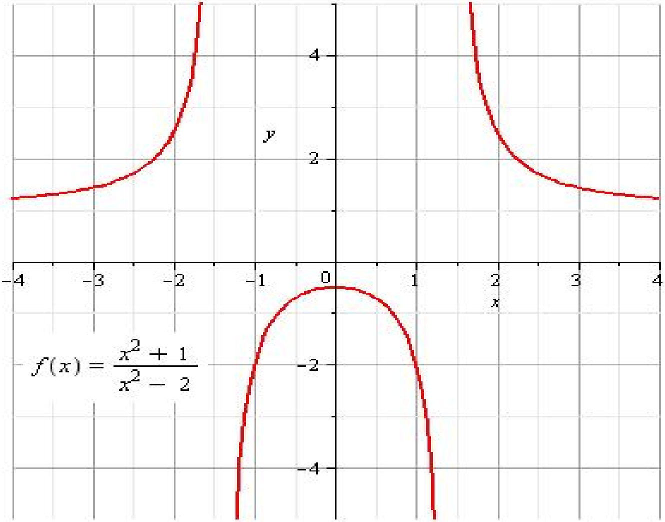


Figure: $f(x) = \frac{x^2+1}{x^2-2}$

Graphs of Rational Functions

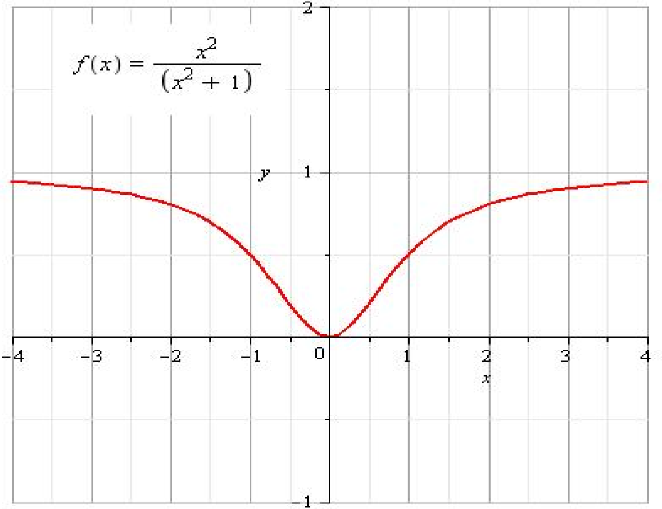


Figure: $f(x) = \frac{x^2}{x^2+1}$

Graphs of Rational Functions

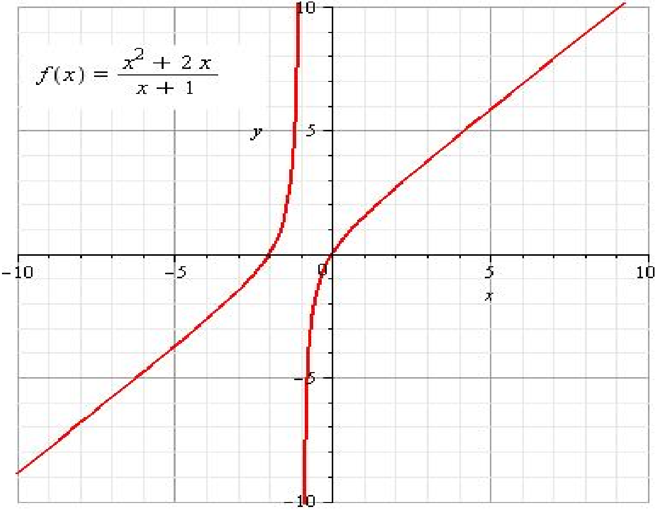


Figure: $f(x) = \frac{x^2+2x}{x+1}$

Graphing Rational Functions: Asymptotes

Unlike a polynomial, a rational function's domain need not be all real numbers, and its graph may have asymptotes.

Question: How does a rational function behave near a value of $x = a$ for which $q(a) = 0$?

It turns out that there are two possibilities:

- The function values **blow up** –i.e go to ∞ or $-\infty$ ($p(a) \neq 0$), or
- the graph has a **hole** in it. ($p(a) = 0$).

Vertical Asymptote

Consider the simple rational function

$$f(x) = \frac{1}{x}.$$

Domain $\{x \mid x \neq 0\}$

Determine the behavior of $f(x)$ for $x \approx 0$.

x	$f(x)$	x	$f(x)$
0.1	10	-0.1	-10
0.01	100	-0.01	-100
0.001	1000	-0.001	-1000
0.0001	10,000	-0.0001	-10,000

Vertical Asymptote

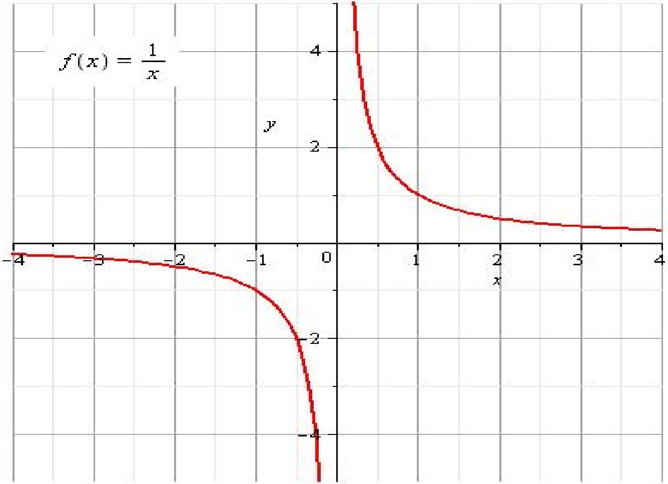


Figure: For $x \approx 0$, the graph seems to hug the vertical line $x = 0$.

Vertical Asymptote

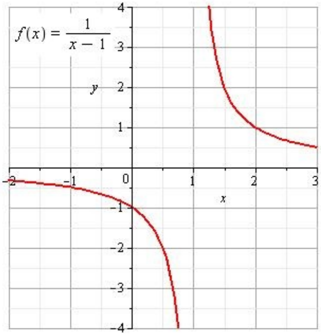
Definition: The vertical line $x = a$ is a vertical asymptote of the function f if $f(x)$ increases without bound as x approaches the number a.

Remark 1: A rational function **MAY** have a vertical asymptote at a value a that is not in its domain.

Remark 2: The graph of a rational function **CANNOT** cross a vertical asymptote.

Some Useful Notation

- $x \to a$ reads as "x approaches a",
- $x \to a^-$ reads as "x approaches a from the left",
- $x \to a^+$ reads as "x approaches a from the right",
- $f(x) \to \infty$ reads as "f of x approaches *infinity*", and
- $f(x) \to -\infty$ reads as "f of x approaches negative *infinity*."



What's in this graph can be described as

$f(x) \to \infty$ as $x \to a^+$

and

$f(x) \to -\infty$ as $x \to a^-$

Lowest Terms

Definition: The rational function

$$f(x) = \frac{p(x)}{q(x)}$$

is in **lowest terms** if the polynomials p and q have no common factors.

Example:

$$f(x) = \frac{(x+2)(x-3)}{x^2(x+5)} \quad \text{is in lowest terms.}$$

$$g(x) = \frac{(x+2)(x-3)}{x^2(x+2)} \quad \text{is NOT in lowest terms.}$$

$$g(x) = \frac{x-3}{x^2} \quad \text{for} \quad x \neq -2$$

Question

Consider the two rational functions F and G.

$$F(x) = \frac{x^2-1}{x^2+x} = \frac{(x-1)(x+1)}{x(x+1)} \qquad G(x) = \frac{x^2-1}{x^2-4} = \frac{(x-1)(x+1)}{(x-2)(x+2)}$$

Which of the following is true?

(a) F and G are in lowest terms.

(b) Neither F nor G is in lowest terms.

(c) F is in lowest terms, but G is not in lowest terms.

(d) F is not in lowest terms, but G is in lowest terms.

Finding Vertical Asymptotes

Theorem: Let $f(x) = \frac{p(x)}{q(x)}$ be a rational function **in lowest terms.** If a is any number such that $q(a) = 0$, then

the line $x = a$ is a vertical asymptote to the graph of f.

Remark: To find vertical asymptote(s): (1) get f into lowest terms, then (2) determine all values that make the denominator zero.

Note that a vertical asymptote is a line—i.e. $x = -2$ *could* be a vertical asymptote, but -2 CANNOT.

Example: $f(x) = \frac{x^2+3x}{x^3-x}$

Determine the domain of the rational function f.

$$f(x) = \frac{x^2+3x}{x(x^2-1)} = \frac{x^2+3x}{x(x-1)(x+1)}$$

The numbers 0, 1, -1 are not in the domain.

The domain is

$$\{x \mid x \neq 0,\ x \neq 1, \text{ and } x \neq -1\}.$$

Example: $f(x) = \frac{x^2+3x}{x^3-x}$

Express f in lowest terms.

$$f(x) = \frac{x(x+3)}{x(x-1)(x+1)}$$

we have a common x

$$f(x) = \frac{x+3}{(x-1)(x+1)} \quad , \quad x \neq 0$$

Example: $f(x) = \frac{x^2+3x}{x^3-x}$

Find the equations of all vertical asymptotes to f.

$$f(x) = \frac{x+3}{(x-1)(x+1)}, \quad x \neq 0$$

There are two vertical asymptotes

$x = -1$ and $x = 1$.

* If $g(x) = \frac{x+3}{(x-1)(x+1)}$, then $g(0) = \frac{0+3}{(0-1)(0+1)} = -3$

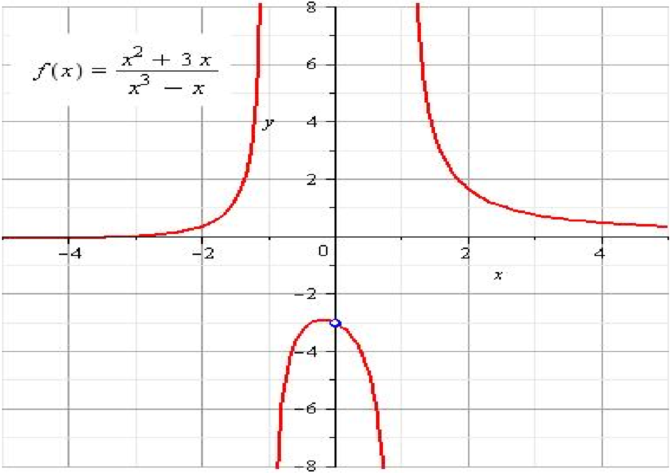


Figure: There is a hole in the graph at the point $(0, -3)$.

Question

Determine the domain of g, and find all vertical asymptotes to g.

$$g(x) = \frac{x^2 - x - 2}{x^2 - 4} = \frac{(x-2)(x+1)}{(x-2)(x+2)}$$

(a) The domain is $(-\infty, -2) \cup (-2, \infty)$, and there is one V. asymptote $x = -2$.

(b) The domain is $\{x \in \mathbb{R} \,|\, x \neq \pm 2\}$, and there is one V. asymptote $x = -2$.

(c) The domain is $\{x \in \mathbb{R} \,|\, x \neq \pm 2\}$, and there are two V. asymptotes $x = -2$ and $x = 2$.

(d) The domain is $(-\infty, -2) \cup (-2, -1) \cup (-1, \infty)$, and there is one V. asymptote $x = -2$.