Sept. 19 Math 1190 sec. 52 Fall 2016

Section 2.4: Differentiating a Product or Quotient; Higher Order Derivatives

Theorem: (Product Rule) Let f and g be differentiable functions of x. Then the product $f(x)g(x)$ is differentiable. Moreover

$$\frac{d}{dx}[f(x)g(x)] = f'(x)g(x) + f(x)g'(x).$$

This can be stated using Leibniz notation as

$$\frac{d}{dx}[f(x)g(x)] = \frac{df}{dx}g(x) + f(x)\frac{dg}{dx}.$$

We used this to find $\frac{d}{dx}e^{2x} = \frac{d}{dx}(e^x e^x) = 2e^{2x}$.

Question

$\frac{d}{dx}(fg) = f'g + fg'$

Use the product rule to evaluate $f'(x)$ where $f(x) = 3x^4e^{2x}$.

(a) $f'(x) = 6x^4e^{2x}$

$f'(x) = 12x^3e^{2x} + 3x^4(2e^{2x})$

(b) $f'(x) = 12x^3e^{2x} + 6x^4e^{2x}$

$= 12x^3e^{2x} + 6x^4e^{2x}$

(c) $f'(x) = 24x^3e^{2x}$

(d) $f'(x) = 3x^4e^{2x} + 12x^3e^{2x}$

The Derivative of a Quotient

Theorem (Quotient Rule) Let f and g be differentiable functions of x. Then on any interval for which $g(x) \neq 0$, the ratio $\frac{f(x)}{g(x)}$ is differentiable. Moreover

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}.$$

This can be stated using Leibniz notation as

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{\frac{df}{dx}g(x) - f(x)\frac{dg}{dx}}{[g(x)]^2}.$$

An immediate consequence of this is that

$$\frac{d}{dx}\left(\frac{1}{g(x)}\right) = -\frac{g'(x)}{[g(x)]^2}.$$

Example

Use the quotient rule to show that for positive integer n*

$$\frac{d}{dx}x^{-n} = -nx^{-n-1}$$

$$\frac{d}{dx}\frac{1}{g(x)} = \frac{-g'(x)}{(g(x))^2}$$

$$\frac{d}{dx}x^{-n} = \frac{d}{dx}\frac{1}{x^n} = -\frac{nx^{n-1}}{(x^n)^2}$$

$$= -\frac{n\cancel{x^n}\cdot x^{-1}}{\cancel{x^n}\cdot x^n} = \frac{-nx^{-1}}{x^n} = -nx^{-1}x^{-n}$$

$$= -nx^{-n-1}$$

The power rule works for all integer powers.

*Note that this shows that the power rule works for both positive and negative integers.

For example

$$\frac{d}{dx} x^{-3} = -3x^{-3-1} = -3x^{-4}$$

Example

$$\frac{d}{dx}\frac{1}{g(x)} = -\frac{g'(x)}{(g(x))^2}$$

Evaluate $\dfrac{d}{dx}e^{-x}$ $= \dfrac{d}{dx}\dfrac{1}{e^x}$

$$= -\frac{e^x}{(e^x)^2}$$

$$= -\frac{\cancel{e^x}}{\cancel{e^x}\cdot e^x} = \frac{-1}{e^x} = -e^{-x}$$

Example

Evaluate $\dfrac{d}{dx}\left(\dfrac{e^x}{x^2+2x}\right)$

$$\frac{d}{dx}\frac{f}{g} = \frac{f'g - fg'}{g^2}$$

Quotient rule

$f(x) = e^x$, $f'(x) = e^x$

$g(x) = x^2+2x$

$g'(x) = 2x+2$

$$= \frac{e^x(x^2+2x) - e^x(2x+2)}{(x^2+2x)^2}$$

$$= \frac{e^x(x^2+2x-2x-2)}{(x^2+2x)^2}$$

$$= \frac{e^x(x^2-2)}{(x^2+2x)^2}$$

Question

Use the quotient rule to evaluate $f'(x)$ where $f(x) = \dfrac{3x+4}{x^2+1}$

$$f'(x) = \frac{3(x^2+1) - (3x+4)(2x)}{(x^2+1)^2}$$

$$= \frac{-3x^2 - 8x + 3}{(x^2+1)^2}$$

(a) $f'(x) = \dfrac{3x^2+8x-3}{(x^2+1)^2}$

(b) $f'(x) = \dfrac{3-2x(3x+4)}{(x^2+1)}$

(c) $f'(x) = \dfrac{-3x^2-8x+3}{(x^2+1)^2}$

(d) $f'(x) = \dfrac{-3x^2-8x+3}{x^4+1}$

Higher Order Derivatives:

Given $y = f(x)$, the function f' may be differentiable as well. We may take its derivative which is called the **second derivative** of f. We use the following notation and language:

$$\frac{d}{dx}\frac{dy}{dx} \sim \frac{ddy}{dx\,dx}$$

First derivative: $\dfrac{dy}{dx} = y' = f'(x)$

Second derivative: $\dfrac{d}{dx}\,\dfrac{dy}{dx} = \dfrac{d^2y}{dx^2} = y'' = f''(x)$

Third derivative: $\dfrac{d}{dx}\,\dfrac{d^2y}{dx^2} = \dfrac{d^3y}{dx^3} = y''' = f'''(x)$

Fourth derivative: $\dfrac{d}{dx}\,\dfrac{d^3y}{dx^3} = \dfrac{d^4y}{dx^4} = y^{(4)} = f^{(4)}(x)$

n^{th} derivative: $\dfrac{d}{dx}\,\dfrac{d^{n-1}y}{dx^{n-1}} = \dfrac{d^ny}{dx^n} = y^{(n)} = f^{(n)}(x)$

Remarks on Notation

- $\frac{d}{dx}$ can *operate* on a function to produce a new function; e.g.

$$\frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d^3y}{dx^3}$$

notice placement of this number

" $\frac{dy^3}{dx^3}$ " doesn't mean anything

- It's too hard to read multiple primes (say beyond 3). Parentheses **must** be used to distinguish powers from derivatives.

$$y^5 \text{ is the fifth power of } y;$$

$$y^{(5)} \text{ is the fifth derivative of } y$$

Example

Compute the first, second, and third derivatives of $f(x) = 3x^4 + 2x^2$.

$$f'(x) = 12x^3 + 4x$$

$$f''(x) = 36x^2 + 4$$

$$f'''(x) = 72x$$

Example

Evaluate $F''(x)$ and $F''(2)$ where $F(x) = x^3 e^x$.

We need F' to find F''

$$\frac{d}{dx} fg = f'g + fg'$$

product rule

$$F'(x) = 3x^2 e^x + x^3 e^x$$

$$F''(x) = 6x e^x + 3x^2 e^x + 3x^2 e^x + x^3 e^x$$

$$= 6x e^x + 6x^2 e^x + x^3 e^x$$

$$= x e^x (6 + 6x + x^2)$$

$$F''(2) = 2e^2\left(6 + 6\cdot 2 + 2^2\right) = 44e^2$$

Questions

(1) Let a, b, and c be nonzero constants. If $y = ax^2 + bx + c$, then $\frac{d^3y}{dx^3}$ is

$\frac{dy}{dx} = 2ax+b$

(a) 0

(b) $2a + b + c$

$\frac{d^2y}{dx^2} = 2a$, $\frac{d^3y}{dx^3} = 0$

(c) $2a$

(d) cannot be determined without knowing the values of a, b, and c.

(2) **True or False:** The fourth derivative of a function $y = f(x)$ is denoted by

$$\frac{dy^4}{dx^4}.$$

false, it's $\frac{d^4y}{dx^4}$

Rectilinear Motion

If the position s of a particle in motion (relative to an origin) is a differentiable function $s = f(t)$ of time t, then the derivatives are physical quantities.

Velocity: is the rate of change of position with respect to time

$$v = \frac{ds}{dt} = f'(t).$$

Acceleration: is the rate of change of velocity with respect to time

$$a = \frac{dv}{dt} = \frac{d^2s}{dt^2} = f''(t).$$

Galileo's Law

Galileo's law states that in a vacuum (i.e. in the absence of fluid drag), the position of any object falling near the Earth's surface, subject only to gravity, is proportional to the square of the time elapsed. Mathematically, position s satisfies

$$s = -ct^2.$$

Show that this statement is equivalent to saying that the acceleration due to gravity is constant.

velocity $v = \frac{ds}{dt} = -2ct$

acceleration $a = \frac{dv}{dt} = -2c$, it's constant

Question

A particle moves along the x-axis so that its position relative to the origin satisfies $s = t^3 - 4t^2 + 5t$. Determine the acceleration of the particle at time $t = 1$.

$v = \frac{ds}{dt} = 3t^2 - 8t + 5$

(a) $a(1) = 0$

(b) $a(1) = -2$

(c) $a(1) = 6t - 8$

(d) $a(1) = 3t^2 - 8t + 5$

$\}$ a(1) is a number

$a = \frac{dv}{dt} = 6t - 8$

$a(1) = 6 \cdot 1 - 8 = -2$

Section 2.5: The Derivative of the Trigonometric Functions

We wish to arrive at derivative rules for each of the six trigonometric functions.

Recall the limits from before

$$\lim_{\theta\to 0}\frac{\sin\theta}{\theta}=1 \quad \text{and} \quad \lim_{\theta\to 0}\frac{\cos\theta-1}{\theta}=0$$

$$\frac{d}{dx}\sin(x) = \cos(x) \quad \text{and} \quad \frac{d}{dx}\cos(x) = -\sin(x)$$

We'll prove the first (the second is left as an exercise).

$$\frac{d}{dx}\sin x = \lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h}$$

$\sin(A+B) = \sin A \cos B + \sin B \cos A$

$$= \lim_{h\to 0} \frac{\sin x \cos h + \sin h \cos x - \sin x}{h}$$

$$= \lim_{h\to 0} \left(\frac{\sin x \cos h - \sin x}{h} + \frac{\sin h \cos x}{h} \right)$$

$$= \lim_{h \to 0} \left[\left(\frac{\cos h - 1}{h} \right) \sin x + \left(\frac{\sin h}{h} \right) \cos x \right]$$

$$= \lim_{h \to 0} \left(\frac{\cos h - 1}{h} \right) \sin x + \lim_{h \to 0} \left(\frac{\sin h}{h} \right) \cos x$$

$$= 0 \cdot \sin x + 1 \cdot \cos x$$

$$= \cos x$$

i.e. $\boxed{\frac{d}{dx} \sin x = \cos x}$

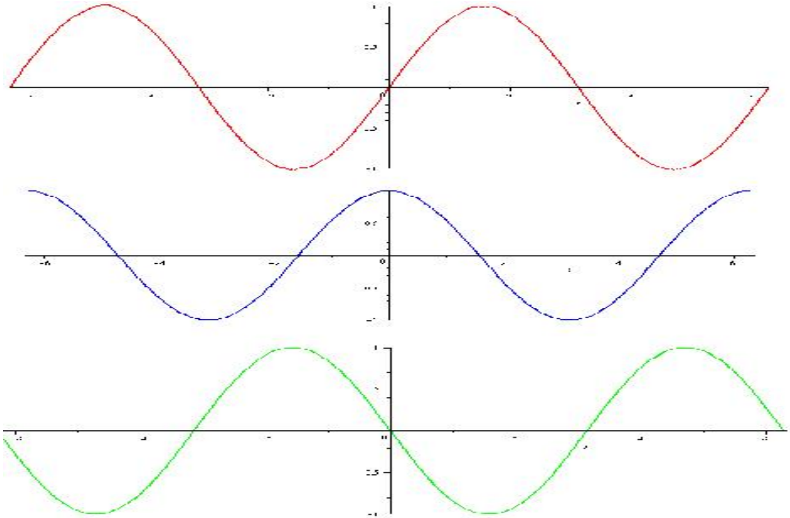

Figure: Graphs of $y = \sin x$, $y = \cos x$, $y = -\sin x$ (from top to bottom).

Examples: Evaluate the derivative.

(a) $\frac{d}{dx}(\sin x + 4\cos x)$ $= \frac{d}{dx}\sin x + 4\frac{d}{dx}\cos x$

$= \cos x + 4(-\sin x) = \cos x - 4\sin x$

(b) $\frac{d}{d\theta}\theta^4 \sin\theta$ $= 4\theta^3 \sin\theta + \theta^4 \cos\theta$

product rule $\frac{d}{dx} fg = f'g + fg'$